Engineering Calculus II - Spring 2016
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QUIZZES AND EXAMS

Week 1 Quiz

Problem 1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x(3-x)$ and $y = 0$ about the line $x = -1$.

Problem 2. Find the exact length of the curve $x = 1 + 4\sin(2t)$, $y = 4\cos(2t) - 3$, $0 \leq t \leq \pi$.

Week 2 Quiz

Problem 1. Find the average value of the function $f(x) = \sin(\pi x)$ on the interval $[-1/2, 1]$.

Problem 2. Find the solution of the differential equation

$$y' = \frac{y}{\sqrt{x-1}}$$

that satisfies the initial condition $y(5) = -3e^4$.

Week 3 Quiz

Problem 1. Determine whether the sequence

$$a_n = \ln\left(\frac{n^2-1}{3+n^2}\right) \quad \text{for } n \geq 2$$

converges or diverges. If it converges, find the limit.

Problem 2. A sequence $\{a_n\}_n$ is given by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{1}{4}(a_n + 6) \quad \text{for } n \geq 1.$$

i) Show that $\{a_n\}_n$ is increasing.
ii) Show that $\{a_n\}_n$ is bounded above by 2.
iii) Determine whether the sequence converges or diverges.

Super Quiz 1

Problem 1. (*6 points.*) Find the value of c such that the following holds

$$\sum_{n=2}^{\infty} 2c^n = 1.$$

Solution. Since the series converges, we have $|c| < 1$. We will solve for c, and we will verify this inequality at the end. We have

$$1 = \sum_{n=2}^{\infty} 2c^n = \sum_{n=0}^{\infty} 2c^n - \sum_{n=0}^{1} 2c^n = \frac{2}{1-c} - 2 - 2c.$$

Hence, we have

$$\frac{2}{1-c} - 2 - 2c = 1.$$

Multiplying by $1-c$ and rearranging, we have $2c^2 + c - 1 = 0$. Solving for c, we find two solutions: $c = -1$ and $c = \frac{1}{2}$. Since we must have $|c| < 1$, the only valid solution is $c = \frac{1}{2}$. □

Problem 2. (*4 points.*) Determine whether the following series converges or diverges; if it converges, find its limit:

$$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{3-k}.$$

Solution. The series can be rewritten as follows

$$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{3-k} = \left(\frac{1}{3}\right)^{3} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{-k} = \left(\frac{1}{3}\right)^{3} \sum_{k=0}^{\infty} 3^k.$$

The series on the right-hand side is a geometric series with common ratio equal to $3 \geq 1$, hence the series diverges. □

Problem 3. (*5 points.*) Determine whether the following series converges or diverges; if it converges, find its limit:

$$\sum_{k=2}^{\infty} \frac{2}{k^2 - 1}.$$

Solution. This is a telescoping series. Using partial fractions decomposition, one has

$$\frac{2}{k^2-1} = \frac{1}{k-1} - \frac{1}{k+1}.$$

Hence, one has

$$\begin{aligned}\sum_{k=2}^{m} \frac{2}{k^2-1} &= \sum_{k=2}^{m}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\\ &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{m-2}-\frac{1}{m}\right)\\ &\quad+\left(\frac{1}{m-1}-\frac{1}{m+1}\right)\\ &= 1+\frac{1}{2}-\frac{1}{m}-\frac{1}{m+1} \xrightarrow{m\to\infty} \frac{3}{2}.\end{aligned}$$

The series thus converges to $\frac{3}{2}$. □

Problem 4. (*5 points.*) Find the solution of the differential equation

$$(x^2+2)y' = xy$$

that satisfies the initial condition $y(0) = 2$.

Solution. Using the method of separation of variables, one has

$$\int \frac{1}{y}dy = \int \frac{x}{x^2+2}dx.$$

Hence, one has

$$\ln|y| = \frac{1}{2}\ln(x^2+2)+C$$

for some constant C. This implies

$$|y| = e^{\frac{1}{2}\ln(x^2+2)+C} = e^{\frac{1}{2}\ln(x^2+2)}\cdot e^{C},$$

and finally

$$y = B\cdot e^{\frac{1}{2}\ln(x^2+2)} = B\cdot\left(e^{\ln(x^2+2)}\right)^{\frac{1}{2}} = B\sqrt{x^2+2}$$

for some constant B. Using the initial condition, one has $2 = y(0) = B\sqrt{2}$, hence $B = \sqrt{2}$. The solution is thus $y(x) = \sqrt{2x^2+4}$. □

Midterm 1

Problem 1. (*12 points.*) A sequence $\{a_n\}_n$ is given by

$$a_1 = 7 \quad\text{and}\quad a_{n+1} = \frac{2}{3}(a_n+2) \quad \text{for } n \geq 1.$$

i) Show that $\{a_n\}_n$ is decreasing.
ii) Show that $\{a_n\}_n$ is bounded below by 4.
iii) Determine whether the sequence converges or diverges.
iv) If the sequence converges, find its limit.

Problem 2. (*12 points.*) Solve the following initial-value problem
$$\frac{y'}{\cos(x)} = -y\cdot\sin^2(x), \quad y(0)=\pi.$$

Solution. Using the method of separation of variables, one has
$$\int \frac{1}{y}dy = -\int \sin^2(x)\cos(x)dx.$$
One deduces
$$\ln|y| = -\frac{\sin^3(x)}{3} + A$$
for some constant A. Solving for y, one has first
$$|y| = e^{-\frac{\sin^3(x)}{3}+A} = B\cdot e^{-\frac{\sin^3(x)}{3}}$$
for some constant B, and then
$$y = C\cdot e^{-\frac{\sin^3(x)}{3}}$$
for some constant C. Using the initial condition, one has $\pi = y(0) = C$, hence the final answer is $y = \pi\cdot e^{-\frac{\sin^3(x)}{3}}$. □

Problem 3. (*12 points.*) Determine whether the following series converges or diverges; if it converges, find its limit
$$\sum_{k=1}^{\infty} \frac{6}{k^2+2k}.$$

Solution. Using partial fractions decomposition, one has
$$\frac{6}{k^2+2k} = \frac{3}{k} - \frac{3}{k+2}.$$
Hence, the partial sums simplify as follows
$$\begin{aligned}\sum_{k=1}^{m}\frac{6}{k^2+2k} &= \sum_{k=1}^{m}\left(\frac{3}{k}-\frac{3}{k+2}\right)\\ &= (3-\cancel{1}) + \left(\frac{3}{2}-\cancel{\frac{3}{4}}\right) + \left(\cancel{1}-\cancel{\frac{3}{5}}\right) + \cdots + \left(\cancel{\frac{3}{m-1}} - \frac{3}{m+1}\right)\\ &\quad + \left(\cancel{\frac{3}{m}} - \frac{3}{m+2}\right)\\ &= 3 + \frac{3}{2} - \frac{3}{m+1} - \frac{3}{m+2} \xrightarrow{m\to\infty} 3 + \frac{3}{2} = \frac{9}{2}.\end{aligned}$$
The series thus converges to $\frac{9}{2}$. □

Problem 4. (*12 points.*) Find the interval of convergence of the following power series

$$\sum_{k=3}^{\infty}(-1)^{k+1}\frac{2^{2k+1}}{\sqrt{2k+1}}(x-1)^k.$$

Problem 5. (*12 points.*) Find a power series representation for the following function, and determine the radius of convergence

$$f(x) = \frac{x^2}{x^3+8}.$$

Problem 6. (**6 Bonus points.**) Find a rational function whose power series representation is equal to the following series

$$\sum_{n=2}^{\infty} n(n-1)x^n.$$

Week 8 Quiz

Problem 1. Find the area of the parallelogram with vertices $A = (0,0)$, $B = (1,4)$, $C = (6,6)$, and $D = (5,2)$.

Problem 2. Find the angle between the vectors $\langle 1,-1,0\rangle$ and $\langle 2,-1,-2\rangle$.

Week 9 Super Quiz

Problem 1. Find the Maclaurin series of the following function, and determine its radius of convergence

$$f(x) = \frac{1}{\sqrt[3]{27-x^3}}.$$

Solution. Using the binomial series, one has

$$\begin{aligned} f(x) &= (27-x^3)^{-1/3} = 27^{-1/3}\left(1-\frac{x^3}{27}\right)^{-1/3} = \frac{1}{3}\sum_{n=0}^{\infty}\binom{-1/3}{n}\left(-\frac{x^3}{27}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{-\frac{1}{3}(-\frac{1}{3}-1)\cdots(-\frac{1}{3}-n+1)}{n!}\frac{x^{3n}}{3^{3n+1}}. \end{aligned}$$

The radius of the binomial series is preserved, hence we have that $|x^3/27| < 1$, which is equivalent to $-3 < x < 3$. The radius is 3. □

Problem 2. Use series to evaluate the following limit

$$\lim_{x\to 0} \frac{1-\cos(2x)}{1-3x-e^{-3x}}.$$

Solution. Since x is approaching 0, we can replace each function with its Maclaurin series. We obtain

$$\lim_{x\to 0} \frac{1-\cos(2x)}{1-3x-e^{-3x}} = \lim_{x\to 0} \frac{1-(1-\frac{4x^2}{2}+\cdots)}{1-3x-(1-3x+\frac{9x^2}{2}+\cdots)} = \lim_{x\to 0} \frac{\frac{4x^2}{2}}{-\frac{9x^2}{2}} = -\frac{4}{9}.$$

□

Problem 3. Find the area of the triangle with vertices $A=(-1,-1)$, $B=(2,3)$, and $C=(5,1)$.

Solution. We can consider the points A, B, and C living on the plan $z=0$. The coordinates will then be $A=(-1,-1,0)$, $B=(2,3,0)$, and $C=(5,1,0)$. The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are now in $\mathbb{R}^3$, and we can take their cross product. The area of the triangle is given by

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{1}{2}|\langle 3,4,0\rangle \times \langle 6,2,0\rangle| = \frac{|-18\vec{k}|}{2} = 9.$$

□

Problem 4. Determine whether the planes $x-y+z=1$ and $y-2z=2$ are parallel, perpendicular, or neither. If they are not parallel, find the line of intersection.

Solution. The normal vector to the plane $x-y+z=1$ is $\vec{n}_1=\langle 1,-1,1\rangle$, and the normal vector to the plane $y-2z=2$ is $\vec{n}_2=\langle 0,1,-2\rangle$. Since the normal vectors $\vec{n}_1$ and $\vec{n}_2$ are not proportional, the planes are not parallel. Since $\vec{n}_1\cdot\vec{n}_2\neq 0$, the two normal vectors are not perpendicular, hence the two planes are not perpendicular. To find the line of intersection, we need a point on the line, and a parallel vector $\vec{v}$.

A point on the line is on both planes, so we need to find a solution of the system

$$\begin{cases} x-y+z=1 \\ y-2z=2 \end{cases}.$$

There are infinitely many solutions to this system (corresponding to the line of intersection). To find one point, we can fix one more condition, like $z=0$, and solve for x and y. We find that the point $(3,2,0)$ lies on both planes, hence lies on the line of intersection.

To find a parallel vector $\vec{v}$, we use the fact that $\vec{v}$ must be orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ (since the line must lie on both planes). Hence, we can take

$$\vec{v}=\vec{n}_1\times\vec{n}_2=\langle 1,2,1\rangle.$$

The line of intersection has vector equation

$$\langle x, y, z\rangle = \langle 3, 2, 0\rangle + t\langle 1, 2, 1\rangle,$$

for $t \in \mathbb{R}$. □

Week 11 Quiz

Problem 1. Find the length of the curve

$$\vec{r}(t) = \langle 4\sin(t), 3t, 4\cos(t)\rangle$$

for $0 \leq t \leq \pi$.

Problem 2. Find the curvature $\kappa(t)$ of the curve

$$\vec{r}(t) = \langle 2 - t, 4t^2, 3 + t\rangle.$$

What is the curvature at the point $(2, 0, 3)$?

Midterm 2

Problem 1. *(12 points.)* Consider the following space curves

$$\begin{aligned} \vec{r}_1(t) &= \langle 1 - t^3, 3, t^2 - 1\rangle, \\ \vec{r}_2(s) &= \langle 2 + s, 1 - 2s, 4 + 5s\rangle. \end{aligned}$$

i) Find the point of intersection of $\vec{r}_1(t)$ and $\vec{r}_2(s)$.

ii) Find the angle of intersection of $\vec{r}_1(t)$ and $\vec{r}_2(s)$.

Problem 2. *(14 points.)* i) Find the equation of the plane that contains the point $(1, 1, 2)$ and the line $x = 2 + t$, $y = 1 - t$, $z = 3t$.

ii) Find the equation of the line passing through the point $(1, 1, 2)$ and meeting the line

$$\vec{r}(t) = \langle 2, 1, 0\rangle + t\langle 1, -1, 3\rangle$$

orthogonally.

Problem 3. *(12 points.)* Consider the curve

$$\vec{r}(t) = \langle \sin(3t), 2t^{\frac{3}{2}}, \cos(3t)\rangle.$$

i) Find the length of the curve $\vec{r}(t)$ for $0 \leq t \leq 3$.

ii) Find the point P on the curve $\vec{r}(t)$ such that the length of the curve $\vec{r}(t)$ between the points $(0, 0, 1)$ and P is 52.

Problem 4. *(12 points.)* Consider the following curve

$$\vec{r}(t) = \langle t^3 - 1, t, 1 - t\rangle.$$

i) Find the curvature $\kappa(t)$ of the curve $\vec{r}(t)$.

ii) Find the point on the curve $\vec{r}(t)$ where the curvature is 0.

Problem 5. *(7 points.)* Sketch the graph of the surface

$$2x^2 - 8x - y^2 + 2y + 7 - 4z = 0.$$

Problem 6. *(7 points.)* Compute the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4 + y^2}.$$

Week 14 Quiz

Problem 1. Let $z(x, y) = \cos(xy)$ with $x = e^s \cdot t$ and $y = t^3$. Use the Chain Rule to find the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $s = 0$ and $t = 1$.

Problem 2. Let $f(x, y) = \cos(xy) + xy$.

i) Find the maximum rate of change of $f(x, y)$ at $(2, 0)$ and the direction in which it occurs.

ii) Find the directional derivative of $f(x, y)$ at $(2, 0)$ in the direction of $\vec{v} = \langle 1, 1\rangle$.

Final Exam

Problem 1. *(4 points.)* Let $f(y)$ be the force in Newtons of gravity upon a rocket with mass m kg as a function of the height y from the Earth's center:

$$f(y) = \frac{GmM}{y^2},$$

where M is the mass of the earth, and G is the gravitational constant. Suppose $GmM =$ 100 N meters-squared.

i) Develop an expression for the work W in Joules that must be consumed to overcome the gravitational force to get the rocket from the Earth's surface $y = 6.3\times10^6$ meters to the low-earth orbit $y = 8.3 \times 10^6$ meters.

ii) Compute your result in (i). You may leave an algebraic expression as an answer.

Problem 2. *(10 points.)* Consider the function $f(x) = \cos(3x)$ on the interval $[-\pi, \pi]$.

i) Compute the Taylor series of $f(x)$ centered at the point $a = 0$.

ii) Compute the second-order Taylor polynomial approximation $T_2(x)$ of $f(x)$ centered at $a = 0$.
iii) Using Taylor's inequality, find a bound for the maximum error $|T_2(x) - f(x)|$ between the approximation $T_2(x)$ and $f(x)$ on the interval $[-\pi, \pi]$.
iv) Suppose that for a practical application, we require that the error does not exceed 1/10 on this interval. Based on your bound in the previous problem, will the second-order Taylor polynomial $T_2(x)$ ensure the required accuracy?

Problem 3. *(8 points.)* Let $G(x, y) = 4 - x^2 - y^2$. Find the points (x, y) that produce the maximum value of G subject to the constraint $F(x, y) = 2xy = 1$.

Problem 4. *(12 points.)* Find the interval of convergence of the following power series

$$\sum_{k=2}^{\infty} (-1)^k \frac{2^{2k}}{3k-1} (x-2)^k.$$

Problem 5. i) *(4 points.)* Find the equation of the tangent plane to $z = x^6 + y^6 - 6xy + 4$ at the point $(0, 1, 5)$.

ii) *(8 points.)* Find the local maximum and minimum values and saddle points of the function

$$f(x, y) = x^6 + y^6 - 6xy + 4.$$

Problem 6. i) *(7 points.)* Find the vector equation of the line of intersection of the planes $x + y - 2z = 2$ and $x - y + 3z = 0$.

ii) *(7 points.)* Find the equation of the plane containing the line of intersection of the planes $x + y - 2z = 2$ and $x - y + 3z = 0$ and perpendicular to the plane $x - 2y + z = 1$.