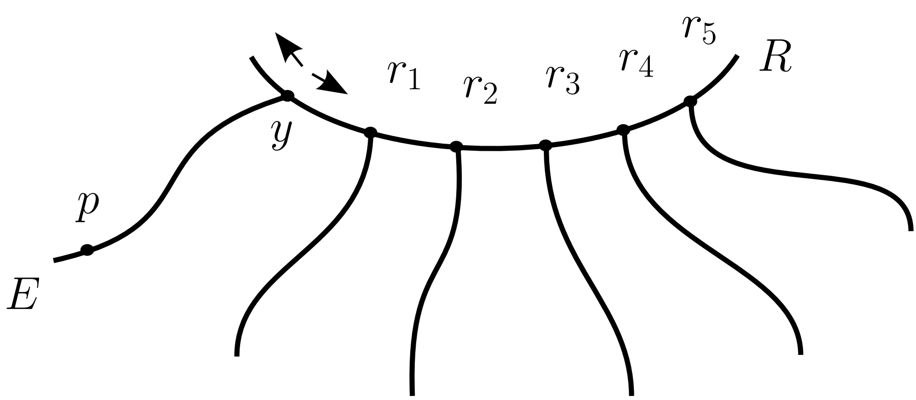
# Geometric cycles in moduli spaces of curves

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# Geometric cycles in moduli spaces of curves

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*Nothing prevents you from eliciting, or as men say learning,*
*out of a single recollection all the rest.*
Plato, *Meno*.

# Abstract

The aim of this thesis is the explicit computation of certain geometric cycles in moduli spaces of curves. In recent years, divisors of $\overline{\mathcal{M}}_{g,n}$ have been extensively studied. Computing classes in codimension one has yielded important results on the birational geometry of the spaces $\overline{\mathcal{M}}_{g,n}$. We give an overview of the subject in Chapter 1.

On the contrary, classes in codimension two are basically unexplored. In Chapter 2 we consider the locus in the moduli space of curves of genus $2k$ defined by curves with a pencil of degree $k$. Since the Brill-Noether number is equal to $-2$, such a locus has codimension two. Using the method of test surfaces, we compute the class of its closure in the moduli space of stable curves.

The aim of Chapter 3 is to compute the class of the closure of the effective divisor in $\mathcal{M}_{6,1}$ given by pointed curves $[C, p]$ with a sextic plane model mapping $p$ to a double point. Such a divisor generates an extremal ray in the pseudoeffective cone of $\overline{\mathcal{M}}_{6,1}$ as shown by Jensen. A general result on some families of linear series with adjusted Brill-Noether number 0 or $-1$ is introduced to complete the computation.

# Zusammenfassung

Ziel dieser Arbeit ist die explizite Berechnung gewisser geometrischer Zykel in Modulräumen von Kurven. In den letzten Jahren wurden Divisoren auf $\overline{\mathcal{M}}_{g,n}$ ausgiebig untersucht. Durch die Berechnung von Klassen in Kodimension 1 konnten wichtige Ergebnisse in der birationalen Geometrie der Räume $\overline{\mathcal{M}}_{g,n}$ erzielt werden. In Kapitel 1 geben wir einen Überblick über dieses Thema.

Im Gegensatz dazu sind Klassen in Kodimension 2 im Großen und Ganzen unerforscht. In Kapitel 2 betrachten wir den Ort, der im Modulraum der Kurven vom Geschlecht $2k$ durch die Kurven mit einem Büschel vom Grad $k$ definiert wird. Da die Brill-Noether-Zahl hier $-2$ ist, hat ein solcher Ort die Kodimension 2. Mittels der Methode der Testflächen berechnen wir die Klasse seines Abschlusses im Modulraum der stabilen Kurven.

Das Ziel von Kapitel 3 ist es, die Klasse des Abschlusses des effektiven Divisors in $\overline{\mathcal{M}}_{6,1}$ zu berechnen, der durch punktierte Kurven $[C, p]$ gegeben ist, für die ein ebenes Modell vom Grad 6 existiert, bei dem $p$ auf einen Doppelpunkt abgebildet wird. Wie Jensen gezeigt hat, erzeugt dieser Divisor einen extremalen Strahl im pseudoeffektiven Kegel von $\overline{\mathcal{M}}_{6,1}$. Ein allgemeines Ergebnis über gewisse Familien von Linearsystemen mit angepasster Brill-Noether-Zahl 0 oder $-1$ wird eingeführt, um die Berechnung zu vervollständigen.

# Acknowledgment

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# Contents

1

# Introduction

Moduli spaces of curves play a central role in classical algebraic geometry. One of their great advantages is a wealth of explicitly described subvarieties. The aim of this thesis is the computation of classes of several interesting loci in codimension one or two. In this chapter we will review some of the main loci known so far and the geometry involved.

For $g \geq 2$, the moduli space $\mathcal{M}_g$ parametrizes smooth complex curves of genus $g$ and has the structure of a Deligne-Mumford stack of dimension $3g-3$. It is compactified by the space $\overline{\mathcal{M}}_g$ of stable curves of genus $g$ and the boundary $\overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ is a divisor with normal crossings. Similarly, for $2g-2+n>0$, the space $\mathcal{M}_{g,n}$ (respectively $\overline{\mathcal{M}}_{g,n}$) parametrizes smooth (respectively stable) $n$-pointed curves of genus $g$ and is a Deligne-Mumford stack of dimension $3g-3+n$.

A natural problem is the study of the properties of $\overline{\mathcal{M}}_{g,n}$ which are birationally invariant. For instance, one would like to compute the Kodaira dimension of the spaces $\overline{\mathcal{M}}_{g,n}$. For a smooth variety $X$, the Kodaira dimension $\kappa(X)$ is defined as the projective dimension of the ring

$$\bigoplus_{n\geq 0} H^0\left(X, nK_X\right)$$

where $K_X$ is the canonical divisor of $X$. Equivalently, $\kappa(X)$ is defined as the largest dimension of the $n$-canonical mapping $\varphi_{nK_X} : X \dashrightarrow \mathbb{P}H^0(X, nK_X)$ for $n \geq 1$. In case of $H^0(X, nK_X) = \emptyset$ for every $n \geq 1$, one sets $\kappa(X) = -\infty$. For a singular variety, one defines the Kodaira dimension to be the Kodaira dimension of any smooth model. The possible values for $\kappa(X)$ are $-\infty, 0, 1, \ldots, \dim X$ and $X$ is said to be of *general type* when $\kappa(X) = \dim X$. For example, for smooth projective varieties in characteristic zero, uniruledness (a property weaker than unirationality or rationally connectedness) implies Kodaira dimension $-\infty$ (see for instance [**Deb01**, Ch. 4 Cor. 4.12]).

A classical result of Severi says that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$, that is, $\kappa(\mathcal{M}_g) = -\infty$ for $g \leq 10$ (see [**AC81a**]). This means that one can describe almost all curves of genus $g \leq 10$ by equations depending on free parameters. Severi conjectured the unirationality of $\overline{\mathcal{M}}_g$ for all $g$. Disproving this conjecture, Harris, Mumford and Eisenbud showed in [**HM82**], [**Har84**] and [**EH87**] that

$\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$, that is, $\kappa(\overline{\mathcal{M}}_g) = \dim \overline{\mathcal{M}}_g = 3g-3$. This is equivalent to say that the canonical class is big, that is, lies in the interior of the pseudoeffective cone.

This result has many important consequences. For instance, when $\kappa(\overline{\mathcal{M}}_g)$ is non-negative, the general curve $C$ of genus $g$ does not admit a polynomial presentation, and if $C$ occurs in a non-trivial linear system on a surface $S$, then $S$ is birational to $C \times \mathbb{P}^1$ ([**HM82**, pg. 26]).

For an effective divisor $D$ in $\overline{\mathcal{M}}_g$, one defines the *slope* of $D$ to be

$$s(D) := \inf \left\{ \frac{a}{b} \text{ for } a, b > 0 \, : \, a\lambda - b\delta - D \equiv \sum_{i=0}^{\lfloor g/2 \rfloor} c_i \delta_i, \text{ where } c_i \geq 0 \; \forall i \right\}$$

(see [**HM90**]). When $D$ is the closure of an effective divisor in $\mathcal{M}_g$, one has that $s(D) < \infty$. In this case, if $D$ has class $a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$, then

$$s(D) = \frac{a}{\min_{i=0}^{\lfloor g/2 \rfloor} b_i}.$$

Harris and Mumford first computed the canonical class of $\overline{\mathcal{M}}_g$

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2 \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i$$

using Kodaira-Spencer theory and the Grothendieck-Riemann-Roch formula.

They showed that pluri-canonical forms defined on the open set of curves without automorphisms extend to any desingularization of $\overline{\mathcal{M}}_g$, hence exhibiting enough global sections of a positive multiple of $K_{\overline{\mathcal{M}}_g}$ gives a lower bound for $\kappa(\overline{\mathcal{M}}_g)$. If there exists an effective divisor $D$ with slope less than $13/2$, the slope of the canonical class, then one has that

$$K_{\overline{\mathcal{M}}_g} \in (13 - 2s(D))\lambda + \frac{2}{\min_{i=0}^{\lfloor g/2 \rfloor} b_i} D + \mathbb{Q}_{\geq 0} \left\langle \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor} \right\rangle$$

where the coefficient of $\lambda$ is positive. It follows that the global sections of $|nK_{\overline{\mathcal{M}}_g}|$ are at least as many as the global sections of $|n(13-2s(D))\lambda|$. Since the class $\lambda$ is big in $\overline{\mathcal{M}}_g$ (see for instance [**Mum77**, Thm. 5.20]), the linear system $|n(13 - 2s(D))\lambda|$ defines a birational morphism to a projective space for sufficiently large $n$, hence $K_{\overline{\mathcal{M}}_g}$ is big as well.

To exhibit a divisor with small slope for odd values of $g = 2k - 1 \geq 25$, Harris and Mumford considered the closure of the Brill-Noether locus of curves admitting a regular map onto $\mathbb{P}^1$ of degree $k$.

## 1.1. Brill-Noether loci in $\mathcal{M}_g$

A linear series $\mathfrak{g}^r_d$ on a smooth projective curve $C$ is a pair $(\mathscr{L}, V)$, where $\mathscr{L} \in \operatorname{Pic}^d(C)$ and $V \subset H^0(C, \mathscr{L})$ is a subvectorspace of dimension $r+1$. Brill-Noether theory studies what $\mathfrak{g}^r_d$ a general curve $[C] \in \mathcal{M}_g$ has. Let us define the *Brill-Noether number*

$$\rho(g, r, d) := g - (r+1)(g - d + r).$$

The following result was introduced by Brill and Noether in [**BN74**]. A rigorous modern proof is due to Griffiths and Harris ([**GH80**]).

THEOREM 1.1.1 (Griffiths-Harris). *A general curve $[C] \in \mathcal{M}_g$ has a linear series $\mathfrak{g}^r_d$ if and only if $\rho(g, r, d) \geq 0$. When so, the variety of linear series $G^r_d(C)$ is pure of dimension $\rho(g, r, d)$.*

It follows that loci of curves carrying a linear series $\mathfrak{g}^r_d$ with negative Brill-Noether number form a subvariety of codimension at least one in $\mathcal{M}_g$. Let $\mathcal{M}^r_{g,d}$ denote the locus in $\mathcal{M}_g$ of curves admitting a $\mathfrak{g}^r_d$. If $\rho(g, r, d) < 0$, then one has that the codimension of $\mathcal{M}^r_{g,d}$ is less than or equal to $-\rho(g, r, d)$ (see [**Ste98**]).

Let us sketch the proof of this result. For a smooth curve $C$ of genus $g$, let $W^r_d(C)$ be the variety parametrizing complete linear series of degree $d$ and dimension at least $r$. The classical description of $W^r_d(C)$ as a determinantal subscheme of $\operatorname{Pic}^n(C)$ for some big enough $n$ (see [**ACGH85**, Ch. VII §2]) extends to the relative situation of smooth families. For a proper smooth family $\pi\colon \mathcal{C} \to S$ of curves of genus $g$, let $\mathcal{P}ic^d(\mathcal{C}/S)$ be the relative Picard variety, that is, the variety parametrizing couples $(C_s, \mathscr{L}_s)$, with $C_s$ being a fiber of $\pi$, $\mathscr{L}_s \in \operatorname{Pic}^d(C_s)$ and $s \in S$. Similarly, let $\mathcal{W}^r_d(\mathcal{C}/S)$ be the variety parametrizing couples $(C_s, \mathscr{L}'_s)$, with $C_s$ being a fiber of $\pi$ and $\mathscr{L}'_s \in W^r_d(C_s)$. Then $\mathcal{W}^r_d(\mathcal{C}/S)$ can be realized as a degeneracy locus in $\mathcal{P}ic^n(\mathcal{C}/S)$ for $n \geq 2g$. This is carried out in [**Ste98**]. There exists a map of vector bundles $\varphi\colon \mathscr{E} \to \mathscr{F}$ over $\mathcal{P}ic^n(\mathcal{C}/S)$ with $\mathscr{E}$ and $\mathscr{F}$ respectively of rank $n-g+1$ and $n-d$, such that $\mathcal{W}^r_d(\mathcal{C}/S)$ is the locus where $\varphi$ has rank less than or equal to $n-g-r$. Being locally defined by the vanishing of all the minors of order $n-g-r+1$ of a $(n-g+1)\times(n-d)$ matrix, $\mathcal{W}^r_d(\mathcal{C}/S)$ has codimension at most

$$[n - g + 1 - (n - g - r)] \cdot [n - d - (n - g - r)] = (r+1)(g - d + r)$$

in $\mathcal{P}ic^n(\mathcal{C}/S)$. Finally, using that $\mathscr{E}$ and $\mathscr{F}$ can be chosen such that $\mathscr{E}^\vee \otimes \mathscr{F}$ is ample relative $f\colon \mathcal{P}ic^n(\mathcal{C}/S) \to S$, Steffen shows that the codimension in $S$ of $f(\mathcal{W}^r_d(\mathcal{C}/S))$, that is, the pull-back of $\mathcal{M}^r_{g,d}$ to $S$ via the moduli map, is at most

$$(r+1)(g-d+r) + \dim S - \dim \mathcal{P}ic^n(\mathcal{C}/S) = -\rho(g, r, d).$$

It follows that, when $\rho(g, r, d)$ is negative, the locus $\mathcal{M}^r_{g,d}$ has codimension less than or equal to $-\rho(g, r, d)$ in $\mathcal{M}_g$.

When $\rho(g,r,d) \in \{-1,-2,-3\}$, one knows that the opposite inequality also holds (see [**EH89**] and [**Edi93**]), hence the locus $\mathcal{M}^r_{g,d}$ is actually pure of codimension $-\rho(g,r,d)$. Moreover, when $r=1$, this is true in any case. Indeed, let $\mathcal{G}^1_d \xrightarrow{\pi} \mathcal{M}_g$ be the variety over $\mathcal{M}_g$ parametrizing couples $(C,l)$ where $C$ is a curve of genus $g$ and $l \in G^1_d(C)$. One has that $\mathcal{G}^1_d$ is smooth of dimension $2g+2d-5$ (see for instance [**AC81b**, pg. 35]) and clearly $\pi(\mathcal{G}^1_d) = \mathcal{M}^1_{g,d}$. B. Segre constructed a smooth curve $C$ of genus $g$ together with a base-point free pencil $l \in G^1_d(C)$ such that the differential of $\pi$ at the point $[(C,l)] \in \mathcal{G}^1_d$ is injective (see for instance [**AC81a**, pg. 346]). Since $\mathcal{M}^1_{g,d}$ is irreducible (see [**Ful69**]), it follows that the dimension of $\mathcal{M}^1_{g,d}$ is $2g+2d-5$, that is, $\mathcal{M}^1_{g,d}$ has codimension exactly $-\rho(g,1,d)$ for every $\rho(g,1,d)<0$.

In [**HM82**] Harris and Mumford considered the locus of curves $\mathcal{M}^1_{2k-1,k}$ in $\mathcal{M}_{2k-1}$ admitting a $\mathfrak{g}^1_k$. Since the Brill-Noether number $\rho(2k-1,1,k)=-1$, this locus is indeed a divisor. Using the method of test curves coupled with admissible covers, they computed the class of its closure in $\overline{\mathcal{M}}_{2k-1}$

$$\left[\overline{\mathcal{M}}^1_{2k-1,k}\right] = \frac{3(2k-4)!}{k!(k-2)!}\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2\rfloor} i(g-i)\delta_i\right) \tag{1.1.1}$$

where $g=2k-1$. This result was later improved in [**EH87**] by Eisenbud and Harris, who computed the class of any Brill-Noether divisor by means of their theory of limit linear series. They showed that the class of $\overline{\mathcal{M}}^r_{g,d}$ with $\rho(g,r,d)=-1$ (hence necessarily $g+1$ composite) is

$$\left[\overline{\mathcal{M}}^r_{g,d}\right] = c_{g,d,r}\cdot \mathcal{BN}$$

for some $c_{g,d,r}>0$, where

$$\mathcal{BN} := (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2\rfloor} i(g-i)\delta_i. \tag{1.1.2}$$

That is, fixing the genus $g$ and varying $r$ and $d$, the classes of all Brill-Noether divisors surprisingly lie in the same ray of the effective cone of $\overline{\mathcal{M}}_g$. In particular, such a ray has slope $<13/2$, hence $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$ and $g+1$ composite. The same result for the remaining values of $g \geq 24$ was obtained by Eisenbud and Harris by considering a divisor of different nature, namely the Gieseker-Petri divisor, see §1.5.

The theory of limit linear series has also been used by Eisenbud and Harris to obtain further results on the subject. For instance, they prove that Brill-Noether divisors are irreducible (see [**EH89**]).

See [**Far09a**] for an account on the state of the art of the Kodaira dimension of $\overline{\mathcal{M}}_g$.

## 1.2. The method of test curves and admissible covers

Harris and Mumford proved that Brill-Noether loci are tautological in $\mathcal{M}_g$. That is, without knowing at the time that the Picard group of $\mathcal{M}_g$ is generated by $\lambda$, they proved that the class of the compactification of a Brill-Noether divisor is equal to an expression of the form

$$\left[\overline{\mathcal{M}}^r_{g,d}\right] = a\lambda - b_0\delta_0 - b_1\delta_1 \cdots - b_{\lfloor g/2\rfloor}\delta_{\lfloor g/2\rfloor}. \tag{1.2.1}$$

Later Harer, Arbarello and Cornalba proved that $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$ is freely generated by $\lambda$ and $\delta_i$'s for $g \geq 3$, hence every divisor is expressible as in (1.2.1) (see [**AC87**]).

The method of test curves consists in intersecting both sides of (1.2.1) with several curves in $\overline{\mathcal{M}}_g$. On one hand, one has to compute the degree of the restrictions of the classes $\lambda$ and $\delta_i$'s to one test curve. On the other hand, one considers the degree of the restriction of the Brill-Noether divisor. For each curve, one thus obtains a linear relation in the coefficients $a$ and $b_i$'s, and varying the curve one can produce enough independent relations to solve the linear system and compute the coefficients.

It is easy to produce curves which are contained in the boundary of $\overline{\mathcal{M}}_g$, hence a good theory of degeneration of linear series is needed. When $r = 1$, this is done by the theory of admissible covers.

The Hurwitz scheme $H_{k,b}$ parametrizes $k$-sheeted coverings $C \to \mathbb{P}^1$ with $b$ ordinary branch points lying over distinct points of $\mathbb{P}^1$, and $C$ a smooth irreducible curve of genus $g$. By the Hurwitz formula, one has $2 - 2g = 2k - b$. The locus $\mathcal{M}^1_{2k-1,k}$ is the image of the map $\varphi\colon H_{k,b} \to \mathcal{M}_{2k-1}$ obtained by forgetting the covering, that is, $\varphi([C \to \mathbb{P}^1]) := [C]$. Harris and Mumford compactified $H_{k,b}$ by the space of admissible covers of degree $k$.

Given a semi-stable curve $C$ of genus $g$ and a stable $b$-pointed curve $(R, p_1, p_2, \ldots, p_b)$ of genus 0, an *admissible cover* is a regular map $\pi\colon C \to B$ such that the followings hold: $\pi^{-1}(B_{\mathrm{smooth}}) = C_{\mathrm{smooth}}$, $\pi|_{C_{\mathrm{smooth}}}$ is simply branched over the points $p_i$ and unramified elsewhere, $\pi^{-1}(B_{\mathrm{singular}}) = C_{\mathrm{singular}}$ and if $C_1$ and $C_2$ are two branches of $C$ meeting at a point $p$, then $\pi|_{C_1}$ and $\pi|_{C_2}$ have same ramification index at $p$. Note that one may attach rational tails at $C$ to cook up the degree of $\pi$.

The above conditions allow to smooth the covering. Thus $[C]$ in $\overline{\mathcal{M}}_{2k-1}$ is in the closure of $\mathcal{M}^1_{2k-1,k}$ if and only if there exists an admissible cover $C' \to R$ of degree $k$ with $C'$ stably equivalent to $C$ (that is, $C$ is obtained from $C'$ by contracting rational components meeting the rest of the curve in at most two points).

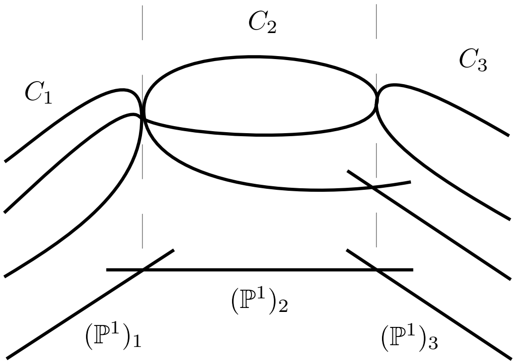


FIGURE 1.2.1. An admissible cover

## 1.3. Enumerative geometry on the general curve

Counting admissible covers (and in general limit linear series) on test curves boils down to solving enumerative problem on the general curve. The first question in this direction is the following. Fix $g, r$ and $d$ such that $\rho(g, r, d) = 0$. By Thm. 1.1.1, the general curve of genus $g$ has only finitely many linear series $\mathfrak{g}^r_d$. How many of them are there?

This problem was elegantly solved by Castelnuovo in [**Cas89**]. His idea was to consider a general singular curve $C$ of arithmetic genus $g$ consisting of a rational curve with $g$ nodes, and count the linear series on this singular curve.

One can realize the normalization $\widetilde{C}$ of $C$ as the rational normal curve $\mathbb{P}^1 \to \mathbb{P}^d$ of degree $d$. If $r_1, \dots, r_g$ are the nodes of $C$, let $p_i, q_i$ be the points in $\widetilde{C}$ that are mapped to the node $r_i$, for $i = 1, \dots, g$. The pull-back to $\widetilde{C}$ of a linear series $\mathfrak{g}^r_d$ on $C$ can then be realized as a linear series on $\widetilde{C}$ swept out by an $r$-dimensional system of hyperplanes with the following property: if one of the hyperplanes contains one of the points $p_i, q_i$, then necessarily contains the other point as well. The intersection of the hyperplanes in one of such systems is a $(d - r - 1)$-plane meeting the lines through $p_i, q_i$. One then counts the $(d-r-1)$-planes with this property.

This is a problem in Schubert calculus. Let $\alpha$ be a Schubert index of type $r, d$, that is, a sequence of integers

$$\alpha : 0 \le \alpha_0 \le \cdots \le \alpha_r \le d - r.$$

One defines

$$\sigma_\alpha \subset \mathrm{Grass}(r + 1, d + 1)$$

to be the variety of $(r + 1)$-planes meeting the $(d + 1 - \alpha_i - i)$-plane of a fixed flag in dimension at least $r + 1 - i$, for $i = 0, \dots, r$. Then the answer to the

Castelnuovo's problem is

$$\sigma^g_{(0,\dots,0,r)} \in H^*(\mathrm{Grass}(d-r, d+1), \mathbb{Z}),$$

that is,

$$N_{g,r,d} := g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}.$$

The next problem is to see what happens when one counts linear series with assigned ramification at a fixed point. Let $C$ be a smooth curve of genus $g$, let $p$ be a point in $C$ and $l$ a $\mathfrak{g}^r_d$ on $C$. To relate $l$ with the point $p$, one considers the *vanishing sequence of $l$ at $p$*

$$a^l(p) : 0 \le a_0 < \cdots < a_r \le d$$

defined as the sequence of distinct orders of vanishing of sections in $V$ at $p$, and the *ramification sequence of $l$ at $p$*

$$\alpha^l(p) : 0 \le \alpha_0 \le \cdots \le \alpha_r \le d - r$$

defined by $\alpha_i := a_i - i$. Let us fix a Schubert index $\alpha$ of type $r, d$.

Theorem 1.3.1 (Eisenbud-Harris). *A general pointed curve $(C, p)$ of genus $g$ admits $l$ a $\mathfrak{g}^r_d$ with $\alpha^l(p) = \alpha$ if and only if*

$$\sum_{i=0}^{r} (\alpha_i + g - d + r)_+ \le g.$$

For the proof we refer to [**EH87**]. Note that the condition in the theorem is stronger than requiring the *adjusted Brill-Noether number*

$$\rho(g, r, d, \alpha) := \rho(g, r, d) - \sum_{i=0}^{r} \alpha_i$$

be non-negative. For instance, while $\rho(1, 2, 4, (0, 2, 2)) = 0$, if $l$ is a $\mathfrak{g}^2_4$ on an elliptic curve $E$ with ramification sequence $\alpha = (0, 2, 2)$ at a point $p \in E$, then $h^0(E, l(-3p)) = 2$, hence $l(-3p)$ produces a $\mathfrak{g}^1_1$ on $E$, a contradiction.

For a general pointed curve $(C, p)$, the variety $G^r_d(C, (p, \alpha))$ of $\mathfrak{g}^r_d$'s with ramification sequence $\alpha$ at the point $p$ is pure of dimension $\rho(g, r, d, \alpha)$. As above, we can study the zero-dimensional case. Suppose $g > 0$ and let $\alpha = (\alpha_0, \dots, \alpha_r)$ be a Schubert index of type $r, d$ such that $\rho(g, r, d, \alpha) = 0$. Then by Thm. 1.3.1, the curve $C$ admits a $\mathfrak{g}^r_d$ with ramification sequence $\alpha$ at the point $p$ if and only if $\alpha_0 + g - d + r \ge 0$. When such linear series exist, there is a finite number of them counted by the *adjusted Castelnuovo number*

$$N_{g,r,d,\alpha} := g! \frac{\prod_{i<j} (\alpha_j - \alpha_i + j - i)}{\prod_{i=0}^{r} (g - d + r + \alpha_i + i)!}. \tag{1.3.1}$$

The idea of the proof is again to specialize to a generic curve of arithmetic genus $g$. One could use a rational spine $R$ with attached $g$ elliptic tails.

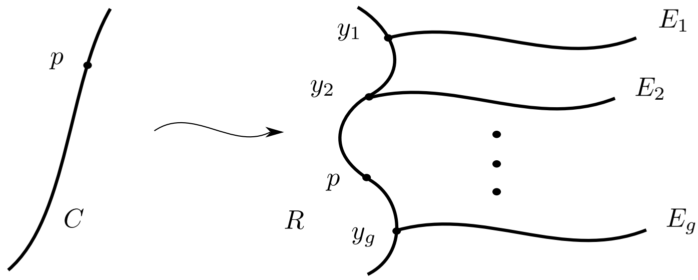


FIGURE 1.3.1. Degeneration of a general pointed curve

Since $p$ is a general point, one can assume that $p$ specializes to $R$. Considering the limit linear series on the singular curve, one sees that the aspects on the elliptic tails are uniquely determined, while the aspect on the rational spine has ramification sequence $\alpha$ at the point $p$ and has ordinary cusps (that is, ramification sequence $(0, 1, \ldots, 1)$) at the points where the elliptic tails are attached. The variety of such linear series on $R$ is reduced, 0-dimensional, and consists of

$$\sigma_\alpha \cdot \sigma^g_{(0,1,\ldots,1)} \in H^*(\mathrm{Grass}(r+1, d-r), \mathbb{Z})$$

points (see [**EH83a**]), whence we obtain (1.3.1) (see [**GH80**, pg. 269]). Note that by duality, the cycle $\sigma_{(0,1,\ldots,1)} \in \mathrm{Grass}(r+1, d-r)$ corresponds to the cycle $\sigma_{(0,\ldots,0,r)} \in \mathrm{Grass}(d-r, d+1)$, and when $\alpha = (0, \ldots, 0)$, we recover the numbers $N_{g,r,d}$.

Instead of considering a fixed general point, one is also interested in the case when the point $p$ is arbitrary. From Thm. 1.3.1, the adjusted Brill-Noether number for a linear series at a general point is necessarily non-negative. On the other hand, the locus of pointed curves admitting a linear series with adjusted Brill-Noether number equal to $-1$ at the marked point is a divisor in $\mathcal{M}_{g,1}$, and when $\rho(g, r, d, \alpha) \leq -2$ then we are in a locus of codimension at least 2 in $\mathcal{M}_{g,1}$ (see [**EH89**]). We deduce that all linear series on a general curve have at any point adjusted Brill-Noether number greater than or equal to $-1$, and imposing the condition $\rho(g, r, d, \alpha) = -1$ singles out a finite number of points.

This problem has been solved by Harris and Mumford for the case $r = 1$. Let $C$ be a general curve of genus $g > 1$ and let $d$ be such that $\rho(g, 1, d) \geq 0$. Since $\rho(g, 1, d, (0, 2d-g-1)) = -1$, there is a finite number of $(x, l_C) \in C \times W^1_d(C)$

such that $\alpha^{l_C}(x) = (0, 2d-g-1)$, that is,

$$n_{g,d,(0,2d-g-1)} := (2d-g-1)(2d-g)(2d-g+1)\frac{g!}{d!(g-d)!}.$$

In order to count such points, one can again use degeneration techniques. Let us sketch the proof. If we consider a rational spine $R$ with attached $g$ elliptic tails $E_1, \ldots, E_g$ respectively at the points $y_1, \ldots, y_g$, then the point $x$ necessarily specializes to one of the elliptic tails. Indeed, let $\pi\colon R \cup E_1 \cup \cdots \cup E_g \to P$ be an admissible cover of degree $d$. If $\pi|_{E_i}$ is unramified at $y_i$, since $E_i$ and $R$ meet only at $y_i$, it follows that $\pi|_{E_i}$ maps $E_i$ with degree one onto a rational curve, a contradiction. Hence $\pi$ has a ramification of order at least two at the points $y_i$ and we require that $x$ be a ramification point of order $2d-g$. If $x$ specializes to $R$, we contradict the Hurwitz formula. Thus $x$ is a smooth point on one of the elliptic tails. There are $g$ possibilities. We can assume that $x \in E_1$. Since $E_1$ and $R$ meet only at $y_1$, necessarily $\pi|_{E_1}$ has the same ramification index at the points $x$ and $y_1$, and up to isomorphism is uniquely determined by the linear series

$$(|(2d-g)x| \cap |(2d-g)y_1|) + (g-d)y_1.$$

By the Hurwitz formula, it follows that $\pi$ is ramified with order exactly 2 at $y_i$, for $i \geq 2$. Thus up to isomorphism, $\pi|_{E_i}$ is uniquely determined by $|2y_i| + (d-2)y_i$ for $i \geq 2$, and there are

$$\sigma_{(0,2d-g-1)} \cdot \sigma_{(0,1)}^{g-1}$$

possibilities for $\pi|_R$. Moreover $x - y_1$ is a non-trivial $(2d-g)$-torsion point in $\mathrm{Pic}^0(E_1)$ and there are $(2d-g)^2 - 1$ such points on $E_1$. Finally there are

$$g[(2d-g)^2 - 1]$$

points with this property on all the elliptic tails and each point counts with multiplicity

$$\sigma_{(0,2d-g-1)} \cdot \sigma_{(0,1)}^{g-1} = (g-1)!\frac{2d-g}{(g-d)!d!}.$$

Similarly when $r > 1$, the problem can be solved using limit linear series.

## 1.4. An example: the closure of the trigonal locus in $\overline{\mathcal{M}}_5$

As an example, we explain how to compute the class of the closure of the *trigonal locus* in $\overline{\mathcal{M}}_5$. That is, we find the coefficients $a, b_0, b_1$ and $b_2$ such that

$$\left[\overline{\mathcal{M}}^1_{5,3}\right] = a\lambda - b_0\delta_0 - b_1\delta_1 - b_2\delta_2 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_5).$$

This is a special case of the formula (1.1.1) and we work it out to clarify the techniques.

**1.4.1. The coefficient $b_1$.** Let $C_4$ be a general curve of genus 4 and let us consider the family of curves $\overline{C}_4$ obtained attaching an elliptic tail $E$ at a moving point $x$ of $C_4$.

The base of this family is then $C_4$. To construct the family, one considers the union of $C_4 \times C_4$ and $C_4 \times E$ and glues together the diagonal $\Delta_{C_4} \subset C_4 \times C_4$ and the constant section $C_4 \times 0_E \subset C_4 \times E$.

This family is entirely contained in the boundary component $\Delta_1$ of $\overline{\mathcal{M}}_5$. All fibers have a unique node of type $\Delta_1$. It follows that on $\overline{C}_4$

$$\deg \delta_1 = \deg \Delta^2_{C_4} + \deg(C_4 \times 0_E)^2 = (2 - 2 \cdot 4) + 0 = -6$$

while the restrictions to $\overline{C}_4$ of all the other generating classes are zero.

On the other hand, let us study the restriction of $\overline{\mathcal{M}}^1_{5,3}$ to $\overline{C}_4$. Let $C_4 \cup_{x \sim 0_E} E$ be the fiber of the family over a point $x \in C_4$ and let us suppose that there exists a degree-3 admissible cover $\pi\colon C_4 \cup_{0_E \sim x} E \to R$. Since $C_4$ is general, $C_4$ is not hyperelliptic while it admits $N_{4,1,3} = 2$ distinct $\mathfrak{g}^1_3$'s, each with 12 points of simple ramification and no other ramification. Then $C_4$ and $E$ are mapped onto two different components of $R$ and the point $x$ is one of the points of simple ramification for a $\mathfrak{g}^1_3$ on $C_4$. Indeed, suppose that $\pi|_{C_4}$ is unramified at $x$. This implies that $\pi|_E$ is also unramified, and since $C_4$ and $E$ meet only in one point, one has that $\pi|_E$ maps the elliptic curve $E$ to a rational curve with degree one, a contradiction. It follows that $\pi|_E$ has degree two and up to isomorphism is uniquely determined by $|2 \cdot 0_E| + 0_E$.

It follows that there are 24 fibers of the family with an admissible cover of degree 3. Since $\overline{C}_4$ is in the interior of $\Delta_1$, such fibers contribute with multiplicity one (see [**HM82**, Thm. 6(b)]). The relation

$$24 = 6b_1$$

follows, whence $b_1 = 4$.

**1.4.2. The coefficient $b_2$.** The procedure to find the coefficient $b_2$ is similar. Let $C_3$ be a general curve of genus 3 and let us consider the family $\overline{C}_3$ obtained identifying a moving point $x$ in $C_3$ with a general point $p$ on a general curve $B$ of genus 2. The base of the family is thus the curve $C_3$. All fibers have a unique node of type $\Delta_2$ and one has that on $\overline{C}_3$

$$\deg \delta_2 = -4$$

while all the other generating classes are zero on this family.

To have an admissible cover on a fiber of the family, one necessarily has that $x$ is a ramification point of order three for some $\mathfrak{g}^1_3$ on $C_3$. Indeed, since $p$ is not a Weiestrass point in $B$, one has that $\pi|_B$ is not of degree 2, and since $B$ and $C_3$ meet only at one point, it follows that $\pi|_B$ is a degree-3 covering with ramification of order three at $p$. Note that up to isomorphism, $\pi|_B$ is uniquely determined by $|3 \cdot p|$.

There are $n_{3,3,(0,2)} = 24$ possibilities for the point $x \in C_3$, hence we obtain the relation

$$24 = 4b_2$$

and we deduce $b_2 = 6$.

**1.4.3. The coefficient $b_0$.** Let $(C_4, p)$ be a general pointed curve of genus 4 and consider the family obtained identifying the point $p$ with a moving point $x$ in $C_4$.

To construct the family, one blows up the surface $C_4 \times C_4$ at the point $(p, p)$ and glue together the proper transform $\widetilde{\Delta}_{C_4}$ of the diagonal $\Delta_{C_4}$ with the proper transform $\widetilde{C_4 \times p}$ of the constant section $C_4 \times p$.

All fibers have a node of type $\Delta_0$. The fiber over $x = p$ has in addition a node of type $\Delta_1$ and the family is smooth at this point. We have that

$$\begin{aligned} \deg \delta_0 &= \deg \widetilde{\Delta}^2_{C_4} + \deg(\widetilde{C_4 \times p})^2 \\ &= \deg \Delta^2_{C_4} - 1 + \deg(C_4 \times p)^2 - 1 \\ &= -8 \\ \deg \delta_1 &= 1 \end{aligned}$$

while the other classes restrict to zero.

Let us consider the intersection with $\overline{\mathcal{M}}^1_{5,3}$. A fiber of the family represents a point in $\overline{\mathcal{M}}^1_{5,3}$ if and only if it has an admissible cover of degree 3 with the points $p$ and $x$ in the same fiber. As in §1.4.1, the curve $C_4$ has two $\mathfrak{g}^1_3$'s, and since the point $p$ is general, we can suppose that $p$ is not a ramification point for any $\mathfrak{g}^1_3$. This automatically excludes the possibility of constructing a desired admissible cover for the fiber over $x = p$. Moreover, we can suppose that in the same fiber of $p$ there are 2 other distinct points for each $\mathfrak{g}^1_3$.

We have found that 4 fibers of the family admit an admissible cover of degree 3, and [**HM82**, Thm. 6(a)] tells us that since such fibers are in the interior of $\Delta_0$, these coverings contribute with multiplicity one.

The relation we get is that

$$4 = 8b_0 - b_1$$

and we recover $b_0 = 1$.

**1.4.4. The coefficient $a$.** The last family is obtained attaching at a general point $p$ on a general curve $C_4$ of genus 4 an elliptic tail varying in a family of elliptic curves $\pi\colon S \to \mathbb{P}^1$ on which the $j$-invariant has degree 12. To construct $S$, one blows up the nine points of intersection of two general plane cubic curves. To construct the desired family of genus 5 curves, identify one of the exceptional divisors $E_0$ with the constant section $\mathbb{P}^1 \times p \subset \mathbb{P}^1 \times C_4$. We have the following restrictions

$$\begin{aligned}\deg \lambda &= \deg \pi_*(\omega_{S/\mathbb{P}^1}) = 1\\ \deg \delta_1 &= \deg E_0^2 + \deg(\mathbb{P}^1 \times p)^2 = -1.\end{aligned}$$

Since the $j$-invariant vanishes at 12 points in $\mathbb{P}^1$, there are 12 nodal rational curves in the family of elliptic curves, and since the family is smooth, we have

$$\deg \delta_0 = 12,$$

while $\deg \delta_2 = 0$.

This family is disjoint from $\overline{\mathcal{M}}^1_{5,3}$. Indeed, reasoning as in 1.4.1, if there exists an admissible cover of degree 3 for a fiber of this family, the point $p$ is necessarily a point of ramification for such a covering, a contradiction since $p$ is general in $C_4$.

We deduce the relation

$$0 = a - 12b_0 + b_1$$

whence we compute the last coefficient $a = 8$.

To summarize we have proved that the class of $\overline{\mathcal{M}}^1_{5,3}$ is

$$\left[\overline{\mathcal{M}}^1_{5,3}\right] = 8\lambda - \delta_0 - 4\delta_1 - 6\delta_2 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_5).$$

## 1.5. The Gieseker-Petri divisor $\mathcal{GP}^1_{(g+2)/2}$

The Gieseker-Petri theorem asserts that for any $l = (\mathscr{L}, V)$ a $\mathfrak{g}^r_d$ on a general curve $C$ of genus $g$, the map

$$\mu_0(l) \colon V \otimes H^0(C, K_C \otimes \mathscr{L}^{-1}) \to H^0(C, K_C) \tag{1.5.1}$$

is injective (see [**EH83b**] or [**Laz86**]). The injectivity of the map $\mu_0(l)$ for a linear series $l$ on an arbitrary curve $C$ of genus $g$ is equivalent to the variety of linear series $G^r_d(C)$ being smooth at the point $l$ and of dimension $\rho(g, r, d)$.

Loci of curves with a linear series failing the Gieseker-Petri condition (1.5.1) then form proper subvarieties of $\mathcal{M}_g$. In particular, for $g = 2(d-1) \geq 4$, the locus $\mathcal{GP}^1_d$ of curves admitting a $\mathfrak{g}^1_d$ failing the Gieseker-Petri condition is a divisor in $\mathcal{M}_g$. It corresponds to the branch locus of the map from the Hurwitz scheme $H_{d,b} \to \mathcal{M}_g$ obtained forgetting the covering and remembering only the source curve. Its class has been computed in [**EH87**]

$$\left[\overline{\mathcal{GP}}^1_d\right] = 2\frac{(2d-4)!}{d!(d-2)!}\left((6d^2 + d - 6)\lambda - d(d-1)\delta_0 - \sum_{i=1}^{g/2} b_i \delta_i\right) \tag{1.5.2}$$

where

$$\begin{aligned} b_1 &= (2d-3)(3d-2) \\ b_2 &= 3(d-2)(4d-3) \end{aligned}$$

while for $3 \leq i \leq d-1$, one has

$$\begin{aligned} b_i &= -(i-2)ib_1 + \frac{(i-1)i}{2}b_2 + (i-2)(i-1)\frac{(g-2)!}{(d-1)!(d-2)!} \\ &\quad - \sum_{l=1}^{\lfloor (i-2)/2 \rfloor} 2(i-1-2l)\frac{(2l)!(g-2-2l)!}{(l+1)!l!(d-l-1)!(d-l)!} \end{aligned}$$

and in particular $b_i > b_{i-1}$. It follows that for even $g \geq 28$, the divisor $\mathcal{GP}^1_d$ has slope less than $13/2$ and this completes Eisenbud and Harris' proof that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

It is known that for $g = 4$ and $g = 6$, the divisor $\overline{\mathcal{GP}}^1_d$ is extremal in the effective cone of $\overline{\mathcal{M}}_g$ (see [**Far10**]). See [**Far09b**] and [**Far10**] for classes of other Gieseker-Petri divisors.

### 1.6. Pointed Brill-Noether divisors in $\mathcal{M}_{g,1}$

As suggested in §1.3, Brill-Noether theory can also produce interesting divisors in the moduli space of pointed curves $\mathcal{M}_{g,1}$. Indeed one can consider the locus $\mathcal{M}^r_{g,d}(\alpha)$ of pointed curves admitting a linear series $\mathfrak{g}^r_d$ with ramification sequence $\alpha$ at the marked point, such that $\rho(g, r, d, \alpha) = -1$. As an example, consider the locus $\mathcal{M}^1_{g,g}(0, g-1)$ of Weierstrass points. Such loci turn out to be useful in the study of the Kodaira dimension of moduli spaces of pointed curves.

Eisenbud and Harris proved that classes of pointed Brill-Noether divisors lie in the two-dimensional cone in the Picard group of $\overline{\mathcal{M}}_{g,1}$ spanned by the pull-back of the class $\mathcal{BN}$ in (1.1.2) of Brill-Noether divisors in $\overline{\mathcal{M}}_g$, that is

$$\mathcal{BN} = (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{g-1} i(g-i)\delta_i$$

and the class of the closure of the locus of Weierstrass points

$$\mathcal{W} := -\lambda + \binom{g+1}{2}\psi - \sum_{i=1}^{g-1} \binom{g-i+1}{2}\delta_i \tag{1.6.1}$$

computed in [**Cuk89**]. It follows that classes of pointed Brill-Noether divisors are expressible as

$$\left[\overline{\mathcal{M}}^r_{g,d}(\alpha)\right] = \mu\mathcal{BN} + \nu\mathcal{W}$$

and $\mu$ and $\nu$ can be computed using test curves. For instance, when $r = 1$, this has been carried out in [**Log03**], where Logan studies the birational geometry of moduli spaces of pointed curves. In general pointed Brill-Noether divisors do not have slope less than the slope of the canonical divisor in $\overline{\mathcal{M}}_{g,1}$. Rather, for non-negative integers $c_1, \ldots, c_n$ with $\sum_i c_i = g > 1$, Logan considers the divisor $D_{g;c_1,\ldots,c_n}$ of pointed curves $(C, p_1, \ldots, p_n)$ in $\overline{\mathcal{M}}_{g,n}$ such that

$$\sum_i c_i p_i$$

moves in a pencil of degree $g$. Letting all the marked points come together, $D_{g;c_1,\ldots,c_n}$ reduces to the Weierstrass divisor in $\overline{\mathcal{M}}_{g,1}$. Logan deduces the class of the divisors of type $D_{g;c_1,\ldots,c_n}$ and applies this to find certain $\overline{n}_g$ such that $\overline{\mathcal{M}}_{g,n}$ is of general type for $n \geq \overline{n}_g$.

## 1.7. Divisors in $\mathcal{M}_{g,n}$ from exceptional secant conditions

Another way of getting effective divisors in $\mathcal{M}_{g,n}$ is to impose exceptional secant conditions at the marked points. For a given linear series $l$ on an arbitrary curve $C$, one can consider the variety of divisors of $C$ that fail to impose independent conditions on $l$.

More precisely, if $l$ is a $\mathfrak{g}^r_d$, we denote by $V^f_e(l)$ the cycle of all divisors $D$ of degree $e$ that impose at most $e-f$ conditions on $l$, that is, $\dim l(-D) \geq r-e+f$ (see [**ACGH85**, Ch. VIII]). The dimension of $V^f_e(l)$ for a general curve $C$ has been computed by Farkas in [**Far08**]. Namely, for a general $l$ in an irreducible component of $G^r_d(C)$, if $V^f_e(l)$ is non-empty, then

$$\dim V^f_e(l) = e - f(r+1-e+f).$$

For example, the variety $V^1_{r+1}(l)$ parametrizes linearly dependent points. If $g, r, d \geq 1$ are such that the Brill-Noether number $\rho(g,r,d) = 0$, then the general curve $C$ of genus $g$ admits a finite number of linear series $l$ of type $\mathfrak{g}^r_d$, and for each of them, the variety $V^1_{r+1}(l)$ of linearly dependent points has dimension $r$. It follows that the following

$$\mathfrak{Lin}^r_d := \{[C, x_1, \ldots, x_{r+1}] \,|\, \exists\, l \text{ a } \mathfrak{g}^r_d \text{ with } x_1 + \cdots + x_{r+1} \in V^1_{r+1}(l)\}$$

is an effective divisor in $\mathcal{M}_{g,r+1}$.

As another example, fix $r = 1$ and choose $g, d \geq 1$. One can consider the variety $V^{n-1}_n(l)$ parametrizing $n$-fold points for $l$ a $\mathfrak{g}^1_d$ on a curve of genus $g$. For a general curve, the variety of $\mathfrak{g}^1_d$'s has dimension $\rho(g,1,d)$, and for each $\mathfrak{g}^1_d$, the variety of $n$-fold points has dimension 1. When $n = \rho(g,1,d)+2$, one has the following divisor in $\mathcal{M}_{g,n}$

$$\mathfrak{Nfold}_{g,d} := \{[C, x_1, \ldots, x_n] \,|\, \exists\, l \text{ a } \mathfrak{g}^1_d \text{ with } x_1 + \cdots + x_n \in V^{n-1}_n(l)\}.$$

The above divisors play an interesting role in the study of the birational geometry of $\mathcal{M}_{g,n}$. Farkas computed the classes of their closures in $\overline{\mathcal{M}}_{g,n}$ and used them to improve the result of Logan, showing that more spaces $\overline{\mathcal{M}}_{g,n}$ are of general type (see [**Far09b**]).

Also note that when $d = g$, the divisor $\mathfrak{Nfold}_{g,g}$ coincide with the Logan's divisor $D_{g;1,\dots,1}$. In this case, Farkas and Verra proved that $\overline{\mathfrak{Nfold}}_{g,g}$ is extremal and rigid in the effective cone of $\overline{\mathcal{M}}_{g,g}$ for $g \leq 11$ (see [**FV09**]).

## 1.8. Outline of the results

While there have been so many works on classes in codimension one, very little is known in higher codimension. The first natural loci in codimension two are Brill-Noether loci. As mentioned in §1.1, when the Brill-Noether number $\rho(g,r,d) = -2$, the locus $\mathcal{M}^r_{g,d}$ has codimension two in $\mathcal{M}_g$. The only class known so far is the class of the closure of the hyperelliptic locus in $\overline{\mathcal{M}}_4$

$$\begin{aligned} 2\left[\overline{\mathcal{M}}^1_{4,2}\right]_Q &= 27\kappa_2 - 339\lambda^2 + 64\lambda\delta_0 + 90\lambda\delta_1 + 6\lambda\delta_2 - \delta_0^2 - 8\delta_0\delta_1 \\ &\quad + 15\delta_1^2 + 6\delta_1\delta_2 + 9\delta_2^2 - 4\delta_{00} - 6\gamma_1 + 3\delta_{01a} - 36\delta_{1,1} \end{aligned} \tag{1.8.1}$$

computed by Faber and Pandharipande in [**FP05**].

In Chapter 2 we compute the class of the closure of the locus $\mathcal{M}^1_{2k,k}$ of curves of genus $2k$ admitting a $\mathfrak{g}^1_k$. For instance when $k = 3$, we obtain the class of the closure of the *trigonal locus* in $\overline{\mathcal{M}}_6$.

THEOREM 1.8.1. *The class of the closure of the trigonal locus in $\overline{\mathcal{M}}_6$ is*

$$\begin{aligned} \left[\overline{\mathcal{M}}^1_{6,3}\right]_Q &= \frac{41}{144}\kappa_1^2 - 4\kappa_2 + \frac{329}{144}\omega^{(2)} - \frac{2551}{144}\omega^{(3)} - \frac{1975}{144}\omega^{(4)} + \frac{77}{6}\lambda^{(3)} \\ &\quad - \frac{13}{6}\lambda\delta_0 - \frac{115}{6}\lambda\delta_1 - \frac{103}{6}\lambda\delta_2 - \frac{41}{144}\delta_0^2 - \frac{617}{144}\delta_1^2 + 18\delta_{1,1} \\ &\quad + \frac{823}{72}\delta_{1,2} + \frac{391}{72}\delta_{1,3} + \frac{3251}{360}\delta_{1,4} + \frac{1255}{72}\delta_{2,2} + \frac{1255}{72}\delta_{2,3} \\ &\quad + \delta_{0,0} + \frac{175}{72}\delta_{0,1} + \frac{175}{72}\delta_{0,2} - \frac{41}{72}\delta_{0,3} + \frac{803}{360}\delta_{0,4} + \frac{67}{72}\delta_{0,5} \\ &\quad + 2\theta_1 - 2\theta_2. \end{aligned}$$

We propose a closed formula for the class of $\overline{\mathcal{M}}^1_{2k,k}$ for $k \geq 3$. Moreover, our computation gives also a new proof of (1.8.1).

In §1.2 we explained how the method of test curves works for divisors' computations. In codimension two one has to use test surfaces. When $g \geq 12$, a basis for the codimension-two rational homology of $\overline{\mathcal{M}}_g$ has been found by Edidin (see [**Edi92**]). Such a basis is composed by tautological classes that are independent for $g \geq 6$. While one does not know a basis for the codimension-two homology of $\overline{\mathcal{M}}_g$ in the range $6 \leq g < 12$, Brill-Noether loci are tautological in $\mathcal{M}_g$. It follows that in any case one has a collection of tautological classes which generate

Brill-Noether classes in codimension two. Intersecting with several test surfaces, we are able to compute the coefficients.

In §1.7 we discussed how to obtain effective divisors in $\mathcal{M}_{g,n}$ by exceptional secant condition. Similarly, one could obtain a codimension two locus in $\mathcal{M}_{g,2}$ and then, pushing it forward to $\mathcal{M}_{g,1}$ by the map that forgets one of the marked points, one obtains a divisor in $\mathcal{M}_{g,1}$. For instance, the general curve of genus 6 admits $N_{6,2,6} = 5$ linear series $\mathfrak{g}^2_6$, and by the Plücker formula, each of them has 4 double points. Hence one has the divisor $\mathfrak{D}^2_6$ of pointed curves $(C, p)$ in $\mathcal{M}_{6,1}$ admitting a sextic plane model mapping $p$ to a double point. In Chapter 3 we compute its class.

THEOREM 1.8.2. *The class of the divisor* $\overline{\mathfrak{D}}^2_6 \subset \overline{\mathcal{M}}_{6,1}$ *is*

$$\left[\overline{\mathfrak{D}}^2_6\right] = 62\lambda + 4\psi - 8\delta_0 - 30\delta_1 - 52\delta_2 - 60\delta_3 - 54\delta_4 - 34\delta_5 \in \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{6,1}).$$

Describing the effective cone of moduli spaces of curves is a notoriously hard problem. Equivalently, one would like to understand all rational contractions of $\overline{\mathcal{M}}_{g,n}$.

Recently Jensen has proven that $\overline{\mathfrak{D}}^2_6$ spans an extremal ray of the effective cone of $\mathcal{M}_{6,1}$ (see [**Jen10**]). The idea is to construct a rational map $\varphi\colon \overline{\mathcal{M}}_{6,1} \dashrightarrow \widetilde{\mathcal{M}}_{0,5}$ as a composition of a birational contraction and a proper morphism using the fact that the canonical model of a general curve of genus 6 is a quadric section of a smooth quintic del Pezzo surface (see also [**SB89**]). Both the divisor $\overline{\mathfrak{D}}^2_6$ and the pull-back from $\overline{\mathcal{M}}_6$ of the Gieseker-Petri divisor $\overline{\mathcal{GP}}^1_4$ map via $\varphi$ to the boundary of $\widetilde{\mathcal{M}}_{0,5}$. Finally Jensen shows how this implies the extremality of $\overline{\mathcal{GP}}^1_4$ and $\overline{\mathfrak{D}}^2_6$ in the effective cone of $\overline{\mathcal{M}}_{6,1}$ by a general property of such a rational map $\varphi$.

The class of the divisor $\overline{\mathcal{GP}}^1_4$

$$\left[\overline{\mathcal{GP}}^1_4\right] = 94\lambda - 12\delta_0 - 50\delta_1 - 78\delta_2 - 88\delta_3 \in \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_6)$$

follows from (1.5.2). Moreover, let us note that in the effective cone of $\overline{\mathcal{M}}_{6,1}$ there is also the Brill-Noether cone spanned by the class of the Weierstrass divisor $\mathcal{W}$ (see (1.6.1)) and the class of the divisor $\overline{\mathcal{M}}^1_{6,4}(0,1)$

$$\left[\overline{\mathcal{M}}^1_{6,4}(0,1)\right] = 15\lambda + 9\psi - 2\delta_0 - 15\delta_1 - 18\delta_2 - 18\delta_3 - 15\delta_4 - 9\delta_5$$

from [**Log03**] (note that it is not known whether the Brill-Noether class $\mathcal{BN}$ (see (1.1.2)) is effective or not in $\overline{\mathcal{M}}_6$). The classes of $\overline{\mathfrak{D}}^2_6$, $\overline{\mathcal{GP}}^1_4$, $\mathcal{W}$ and $\overline{\mathcal{M}}^1_{6,4}(0,1)$ span a 4-dimensional cone. The complete effective cone of $\overline{\mathcal{M}}_{6,1}$ is unknown.

The following two chapters are based respectively on the following papers:

- ◇ N. Tarasca, *Brill-Noether loci in codimension two*,
- ◇ N. Tarasca, *Double points of plane models in* $\overline{\mathcal{M}}_{6,1}$, Jour. Pure and App. Alg. 216 (2012), pp. 766–774.

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# Brill-Noether loci in codimension two

The classical Brill-Noether theory is of crucial importance for the geometry of moduli of curves. While a general curve admits only linear series with non-negative Brill-Noether number, the locus $\mathcal{M}^r_{g,d}$ in $\mathcal{M}_g$ of curves of genus $g$ admitting a $\mathfrak{g}^r_d$ with Brill-Noether number $\rho(g,r,d)=-1$ is a codimension-one subvariety. Eisenbud, Harris and Mumford have extensively studied such a locus. They computed its class and found that it has slope $6+12/(g+1)$. Since for $g \geq 24$ this is less than $13/2$ the slope of the canonical bundle, it follows that $\overline{\mathcal{M}}_g$ is of general type for $g$ composite and greater than or equal to 24.

While in recent years classes of divisors in $\overline{\mathcal{M}}_g$ have been extensively investigated, codimension-two subvarieties are basically unexplored. A natural candidate is offered from Brill-Noether theory. Since $\rho(2k,1,k)=-2$, the locus $\mathcal{M}^1_{2k,k} \subset \mathcal{M}_{2k}$ of curves of genus $2k$ admitting a pencil of degree $k$ has codimension two (see [**Ste98**]). As an example, consider the hyperelliptic locus $\mathcal{M}^1_{4,2}$ in $\mathcal{M}_4$.

Our main result is the explicit computation of the classes of the closures of such loci. When $g \geq 12$, a basis for the codimension-two rational homology of the moduli space $\overline{\mathcal{M}}_g$ of stable curves has been found by Edidin ([**Edi92**]). It consists of the tautological classes $\kappa_1^2$ and $\kappa_2$ together with boundary classes. While such classes are still linearly independent for $g \geq 6$, there might be further generating classes coming from the interior of $\overline{\mathcal{M}}_g$ for $6 \leq g < 12$. This problem can be overcome since one knows that Brill-Noether loci lie in the tautological ring of $\mathcal{M}_g$.

Indeed in a similar situation, Harris and Mumford computed classes of Brill-Noether divisors in $\overline{\mathcal{M}}_g$ before knowing that $\mathrm{Pic}(\mathcal{M}_g)$ is generated solely by the class $\lambda$, by showing that such classes lie in the tautological ring of $\mathcal{M}_g$ (see [**HM82**, Thm. 3]). Their argument works in arbitrary codimension. In our case, since $r=1$, one could alternatively use a result of Faber and Pandharipande which says that the loci of type $\overline{\mathcal{M}}^1_{g,d}$ are tautological in $\overline{\mathcal{M}}_g$ ([**FP05**]).

Having then a basis for the classes of Brill-Noether codimension-two loci, in order to determine the coefficients we use the method of test surfaces. That is, we produce several surfaces in $\overline{\mathcal{M}}_g$ and, after evaluating the intersections on one hand with the classes in the basis and on the other hand with the Brill-Noether loci, we obtain enough independent relations in order to compute the coefficients of the sought-for classes.

The surfaces used are bases of families of curves with several nodes, hence a good theory of degeneration of linear series is required. For this, the compactification of the Hurwitz scheme by the space of admissible covers introduced by Harris and Mumford comes into play. The intersection problems thus boil down first to counting pencils on the general curve, and then to evaluating the respective multiplicities via a local study of the compactified Hurwitz scheme.

For instance when $k = 3$, we obtain the class of the closure of the trigonal locus in $\overline{\mathcal{M}}_6$.

Theorem 2.0.3. *The class of the closure of the trigonal locus in $\overline{\mathcal{M}}_6$ is*

$$\begin{aligned}\left[\overline{\mathcal{M}}^1_{6,3}\right]_Q \quad = \quad & \frac{41}{144}\kappa_1^2 - 4\kappa_2 + \frac{329}{144}\omega^{(2)} - \frac{2551}{144}\omega^{(3)} - \frac{1975}{144}\omega^{(4)} + \frac{77}{6}\lambda^{(3)} \\ & - \frac{13}{6}\lambda\delta_0 - \frac{115}{6}\lambda\delta_1 - \frac{103}{6}\lambda\delta_2 - \frac{41}{144}\delta_0^2 - \frac{617}{144}\delta_1^2 + 18\delta_{1,1} \\ & + \frac{823}{72}\delta_{1,2} + \frac{391}{72}\delta_{1,3} + \frac{3251}{360}\delta_{1,4} + \frac{1255}{72}\delta_{2,2} + \frac{1255}{72}\delta_{2,3} \\ & + \delta_{0,0} + \frac{175}{72}\delta_{0,1} + \frac{175}{72}\delta_{0,2} - \frac{41}{72}\delta_{0,3} + \frac{803}{360}\delta_{0,4} + \frac{67}{72}\delta_{0,5} \\ & + 2\theta_1 - 2\theta_2.\end{aligned}$$

We produce a closed formula for the class of $\overline{\mathcal{M}}^1_{2k,k}$ verified for $3 \le k \le 100$ and we conjecture its validity for every $k \ge 3$.

Theorem 2.0.4. *For $3 \le k \le 100$ the class of the locus $\overline{\mathcal{M}}^1_{2k,k}$ in $\overline{\mathcal{M}}_{2k}$ is*

$$\begin{aligned}\left[\overline{\mathcal{M}}^1_{2k,k}\right]_Q \quad = \quad & \frac{2^{k-6}(2k-7)!!}{3(k!)}\Bigg[(3k^2+3k+5)\kappa_1^2 - 24k(k+5)\kappa_2 \\ & + \sum_{i=2}^{2k-2}\Big( - 180i^4 + 120i^3(6k+1) - 36i^2\left(20k^2+24k-5\right) \\ & + 24i\left(52k^2-16k-5\right) + 27k^2 + 123k + 5\Big)\omega^{(i)} + \cdots\Bigg].\end{aligned}$$

The complete formula is shown in §2.7. We also test our result in several ways, for example by pulling-back to $\overline{\mathcal{M}}_{2,1}$. The computations include the case $g = 4$ which was previously known: the hyperelliptic locus in $\overline{\mathcal{M}}_4$ has been computed in [**FP05**, Prop. 5].

## 2.1. A basis for $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$

In [**Edi92**] Edidin gives a basis for the codimension-two rational homology of $\overline{\mathcal{M}}_g$ for $g \geq 12$. Let us quickly recall the notation. There are the tautological classes $\kappa_1^2$ and $\kappa_2$ coming from the interior $\mathcal{M}_g$; the following products of classes from $\mathrm{Pic}(\overline{\mathcal{M}}_g)$: $\lambda\delta_0, \lambda\delta_1, \lambda\delta_2, \delta_0^2$ and $\delta_1^2$; the following push-forwards $\lambda^{(i)}, \lambda^{(g-i)}, \omega^{(i)}$ and $\omega^{(g-i)}$ of the classes $\lambda$ and $\omega$ respectively from $\mathcal{M}_{i,1}$ and $\mathcal{M}_{g-i,1}$ to $\Delta_i$: $\lambda^{(3)}, \ldots, \lambda^{(g-3)}$ and $\omega^{(2)}, \ldots, \omega^{(g-2)}$; for $1 \leq i \leq \lfloor (g-1)/2 \rfloor$ the $Q$-class $\theta_i$ of the closure of the locus $\Theta_i$ whose general element is a union of a curve of genus $i$ and a curve of genus $g-i-1$ attached at two points; finally the classes $\delta_{ij}$ defined as follows. The class $\delta_{00}$ is the $Q$-class of the closure of the locus $\Delta_{00}$ whose general element is an irreducible curve with two nodes. For $1 \leq j \leq g-1$ the class $\delta_{0j}$ is the $Q$-class of the closure of the locus $\Delta_{0j}$ whose general element is an irreducible nodal curve of geometric genus $g-j-1$ together with a tail of genus $j$. At last for $1 \leq i \leq j \leq g-2$ and $i+j \leq g-1$, the class $\delta_{ij}$ is defined as $\delta_{ij} := [\overline{\Delta}_{ij}]_Q$, where $\Delta_{ij}$ has as general element a chain of three irreducible curves with the external ones having genus $i$ and $j$.

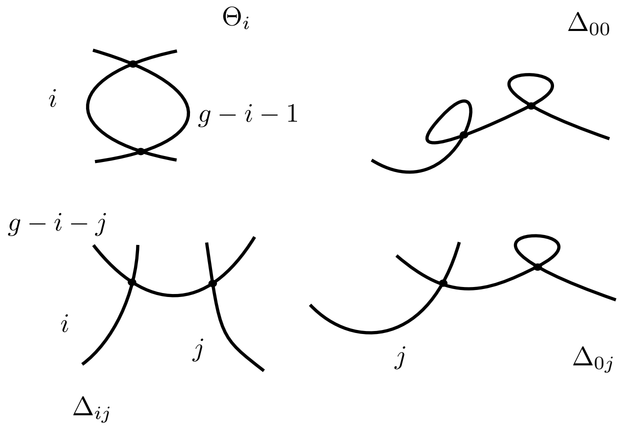



FIGURE 2.1.1. Loci in $\overline{\mathcal{M}}_g$

Finally the codimension-two rational homology of $\overline{\mathcal{M}}_g$ for $g \geq 12$ has rank

$$\lfloor (g^2-1)/4 \rfloor + 3g - 1.$$

Moreover Edidin's proof shows that the above classes are linearly independent also for $6 \leq g < 12$. While in this range there might be other generators coming from the interior of $\overline{\mathcal{M}}_g$, using an argument similar to [**HM82**, Thm. 3] one knows that the class of $\mathcal{M}^1_{2k,k}$ lies in the tautological ring. Hence in any case, for

$g = 2k \geq 6$ we can write

$$\begin{aligned}\left[\overline{\mathcal{M}}^1_{2k,k}\right]_Q &= A_{\kappa_1^2}\kappa_1^2 + A_{\kappa_2}\kappa_2 + A_{\delta_0^2}\delta_0^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\delta_1^2}\delta_1^2 + A_{\lambda\delta_1}\lambda\delta_1 \\ &\quad + A_{\lambda\delta_2}\lambda\delta_2 + \sum_{i=2}^{g-2} A_{\omega^{(i)}}\omega^{(i)} + \sum_{i=3}^{g-3} A_{\lambda^{(i)}}\lambda^{(i)} + \sum_{i,j} A_{\delta_{ij}}\delta_{ij} \\ &\quad + \sum_{i=1}^{\lfloor (g-1)/2\rfloor} A_{\theta_i}\theta_i\end{aligned} \tag{2.1.1}$$

in $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$.

## 2.2. On the method of test surfaces

The method of test surfaces has been developed in [**Edi92**]. See [**Edi92**, §3.1.2, §3.4 and Lemma 4.3] for computing the restriction of the generating classes to cycles parametrizing curves with nodes. In this section we note how to compute the restriction of the class $\kappa_2$ and the classes $\omega^{(i)}$ and $\lambda^{(i)}$ to the test surfaces (S1) - (S14) in the next section.

PROPOSITION 2.2.1. *Let $\pi_1 \colon X_1 \to B_1$ be a one-dimensional family of stable curves of genus $i$ with section $\sigma_1 \colon B_1 \to X_1$ and similarly let $\pi_2 \colon X_2 \to B_2$ be a one-dimensional family of stable curves of genus $g-i$ with section $\sigma_2 \colon B_2 \to X_2$. Obtain a two-dimensional family of stable curves $\pi \colon X \to B_1 \times B_2$ as the union of $X_1 \times B_2$ and $B_1 \times X_2$ modulo glueing $\sigma_1(B_1) \times B_2$ with $B_1 \times \sigma_2(B_2)$. Then the class $\kappa_2$ and the classes $\omega^{(i)}$ and $\lambda^{(i)}$ restrict to $B_1 \times B_2$ as follows*

$$\begin{array}{rcll}\kappa_2 &=& 0 & \\ \omega^{(i)} = \omega^{(g-i)} &=& -\pi_{1*}(\sigma_1^2(B_1))\pi_{2*}(\sigma_2^2(B_2)) & \textit{if } 2 \leq i < g/2 \\ \omega^{(g/2)} &=& -2\pi_{1*}(\sigma_1^2(B_1))\pi_{2*}(\sigma_2^2(B_2)) & \textit{if } g \textit{ is even} \\ \omega^{(j)} &=& 0 & \textit{for } j \notin \{i, g-i\} \\ \lambda^{(i)} &=& \lambda_{B_1}\pi_{2*}(\sigma_2^2(B_2)) & \textit{if } 3 \leq i < g/2 \\ \lambda^{(g-i)} &=& \lambda_{B_2}\pi_{1*}(\sigma_1^2(B_1)) & \textit{idem} \\ \lambda^{(g/2)} &=& \lambda_{B_1}\pi_{2*}(\sigma_2^2(B_2)) + \lambda_{B_2}\pi_{1*}(\sigma_1^2(B_1)) & \textit{if } g \textit{ is even} \\ \lambda^{(j)} &=& \lambda_{B_1}\delta_{j-i,1}|_{B_2} + \lambda_{B_2}\delta_{j-g+i,1}|_{B_1} & \textit{for } j \notin \{i, g-i\}\end{array}$$

*where $\delta_{h,1}|_{B_1} \in \mathrm{Pic}(\overline{\mathcal{M}}_{i,1})$ and similarly $\delta_{h,1}|_{B_2} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g-i,1})$.*

PROOF. Let $\nu \colon \widetilde{X} \to X$ be the normalization, where $\widetilde{X} := X_1 \times B_2 \cup B_1 \times X_2$. Let $K_{X/B_1\times B_2} = c_1(\omega_{X/B_1\times B_2})$. We have

$$\begin{aligned}\kappa_2 &= \pi_*\left(K^3_{X/B_1\times B_2}\right) \\ &= \pi_*\nu_*\left((\nu^* K_{X/B_1\times B_2})^3\right)\end{aligned}$$

where we have used that $\nu$ is a proper morphism, hence the push-forward is well-defined. One has

$$K_{\widetilde{X}/B_1\times B_2} = \left(K_{X_1/B_1} \times B_2\right) \oplus \left(B_1 \times K_{X_2/B_2}\right)$$

hence

$$\nu^* K_{X/B_1\times B_2} = \left((K_{X_1/B_1} + \sigma_1(B_1)) \times B_2\right) \oplus \left(B_1 \times (K_{X_2/B_2} + \sigma_2(B_2))\right).$$

Finally

$$\left((K_{X_1/B_1} + \sigma_1(B_1)) \times B_2\right)^3 = (K_{X_1/B_1} + \sigma_1(B_1))^3 \times B_2 = 0$$

since $K_{X_1/B_1} + \sigma_1(B_1)$ is a class on the surface $X_1$, and similarly for $B_1 \times (K_{X_2/B_2} + \sigma_2(B_2))$, hence $\kappa_2$ is zero.

The statement about the classes $\omega^{(i)}$ and $\lambda^{(i)}$ follows almost by definition. For instance, since the divisor $\delta_i$ is

$$\delta_i = \pi_*(\sigma_1^2(B_1) \times B_2) + \pi_*(B_1 \times \sigma_2^2(B_2))$$

we have

$$\omega^{(i)} \quad = \quad -\pi_{1*}(\sigma_1^2(B_1)) \cdot \pi_{2*}(\sigma_2^2(B_2)).$$

The other equalities follow in a similar way. $\square$

Moreover Mumford's formula for $\kappa_1$ will be constantly used: if $g > 1$ then

$$\kappa_1 = 12\lambda - \delta$$

in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ (see [**Mum77**]).

## 2.3. Enumerative geometry on the general curve

Let $C$ be a complex smooth projective curve of genus $g$ and $l = (\mathscr{L}, V)$ a linear series of type $\mathfrak{g}^r_d$ on $C$, that is $\mathscr{L} \in \mathrm{Pic}^d(C)$ and $V \subset H^0(C, \mathscr{L})$ is a subspace of vector-space dimension $r+1$. The *vanishing sequence* $a^l(p) : 0 \leq a_0 < \cdots < a_r \leq d$ of $l$ at a point $p \in C$ is defined as the sequence of distinct order of vanishing of sections in $V$ at $p$, and the *ramification sequence* $\alpha^l(p) : 0 \leq \alpha_0 \leq \cdots \leq \alpha_r \leq d - r$ as $\alpha_i := a_i - i$, for $i = 0, \ldots, r$. The *weight* $w^l(p)$ will be the sum of the $\alpha_i$'s.

Given an $n$-pointed curve $(C, p_1, \ldots, p_n)$ of genus $g$ and $l$ a $\mathfrak{g}^r_d$ on $C$, the *adjusted Brill-Noether number* is

$$\begin{aligned} \rho(C, p_1, \ldots p_n) \quad &= \quad \rho(g, r, d, \alpha^l(p_1), \ldots, \alpha^l(p_n)) \\ &:= \quad g - (r+1)(g - d + r) - \sum_{i,j} \alpha^l_j(p_i). \end{aligned}$$

**2.3.1. Fixing two general points.** Let $(C,p,q)$ be a general 2-pointed curve of genus $g \geq 1$ and let $\alpha = (\alpha_0,\dots,\alpha_r)$ and $\beta = (\beta_0,\dots,\beta_r)$ be Schubert indices of type $r,d$ (that is $0 \leq \alpha_0 \leq \cdots \leq \alpha_r \leq d-r$ and similarly for $\beta$) such that $\rho(g,r,d,\alpha,\beta) = 0$. The number of linear series $\mathfrak{g}^r_d$ having ramification sequence $\alpha$ at the point $p$ and $\beta$ at the point $q$ is counted by the *adjusted Castelnuovo number*

$$g!\det\left(\frac{1}{[\alpha_i + i + \beta_{r-j} + r - j + g - d]!}\right)_{0\leq i,j\leq r}$$

where $1/[\alpha_i + i + \beta_{r-j} + r - j + g - d]!$ is taken to be 0 when the denominator is negative (see §1.3 and [**Ful98**, Ex. 14.7.11 (v)]). Note that the above expression may be zero, that is the set of sought linear series may be empty.

When $r = 1$ let us denote by $N_{g,d,\alpha,\beta}$ the above expression. If $\alpha_0 = \beta_0 = 0$ then

$$N_{g,d,\alpha,\beta} = g!\left(\frac{1}{(\beta_1+1+g-d)!(\alpha_1+1+g-d)!} - \frac{1}{(g-d)!(\alpha_1+\beta_1+2+g-d)!}\right).$$

Subtracting the base locus $\alpha_0 p + \beta_0 q$, one can reduce the count to the case $\alpha_0 = \beta_0 = 0$, hence $N_{g,d,\alpha,\beta} = N_{g,d-\alpha_0-\beta_0,(0,\alpha_1-\alpha_0),(0,\beta_1-\beta_0)}$.

In the following we will also use the abbreviation $N_{g,d,\alpha}$ when $\beta$ is zero, that is $N_{g,d,\alpha}$ counts the linear series with the only condition of ramification sequence $\alpha$ at a single general point.

**2.3.2. A moving point.** Let $C$ be a general curve of genus $g > 1$ and $\alpha = (\alpha_0,\alpha_1)$ be a Schubert index of type $1,d$ (that is $0 \leq \alpha_0 \leq \alpha_1 \leq d-1$). When $\rho(g,1,d,\alpha) = -1$, there is a finite number $n_{g,d,\alpha}$ of $(x, l_C) \in C \times W^1_d(C)$ such that $\alpha^{l_C}(x) = \alpha$. (Necessarily $\rho(g,1,d) \geq 0$ since the curve is general.) Assuming $\alpha_0 = 0$, one has $\alpha_1 = 2d - g - 1$ and

$$n_{g,d,\alpha} = (2d-g-1)(2d-g)(2d-g+1)\frac{g!}{d!(g-d)!}.$$

If $\alpha_0 > 0$ then $n_{g,d,\alpha} = n_{g,d-\alpha_0,(0,\alpha_1-\alpha_0)}$. Each $\tilde{l}_C := l_C(-\alpha_0 x)$ satisfies $h^0(\tilde{l}_C) = 2$, is generated by global sections, and $H^0(C,\tilde{l}_C)$ gives a covering of $\mathbb{P}^1$ with ordinary brach points except for a $(\alpha_1-\alpha_0)$-fold branch point, all lying over distinct points of $\mathbb{P}^1$. Moreover, since for general $C$ the above points $x$ are distinct, one can suppose that fixing one of them, the $l_C$ is unique. See [**HM82**, Thm. B and pg. 78]. Clearly $\alpha$ in the lower indexes of the numbers $n$ is redundant in our notation, but for our purposes it is useful to keep track of it.

**2.3.3. Two moving points.** Let $C$ be a general curve of genus $g > 1$ and $\alpha = (\alpha_0, \alpha_1)$ be a Schubert index of type $1, d$ (that is $0 \leq \alpha_0 \leq \alpha_1 \leq d-1$). When $\rho(g, 1, d, \alpha, (0, 1)) = -2$ (and $\rho(g, 1, d) \geq 0$), there is a finite number $m_{g,d,\alpha}$ of $(x, y, l_C) \in C \times C \times G^1_d(C)$ such that $\alpha^{l_C}(x) = \alpha$ and $\alpha^{l_C}(y) = (0, 1)$. Subtracting the base locus as usual, one can always reduce to the case $\alpha_0 = 0$.

LEMMA 2.3.1. *Assuming $\alpha_0 = 0$, one has that*

$$m_{g,d,\alpha} = n_{g,d,\alpha} \cdot (3g - 1)\,.$$

PROOF. Since $\rho(g, 1, d, \alpha) = -1$, one can compute first the number of points of type $x$, and then fixing one of these, use the Riemann-Hurwitz formula to find the number of points of type $y$. □

## 2.4. Compactified Hurwitz scheme

Let $H_{k,b}$ be the Hurwitz scheme parametrizing coverings $\pi\colon C \to \mathbb{P}^1$ of degree $k$ with $b$ ordinary branch points and $C$ a smooth irreducible curve of genus $g$. By considering only the source curve $C$, $H_{k,b}$ admits a map to $\mathcal{M}_g$

$$\sigma\colon H_{k,b} \to \mathcal{M}_g.$$

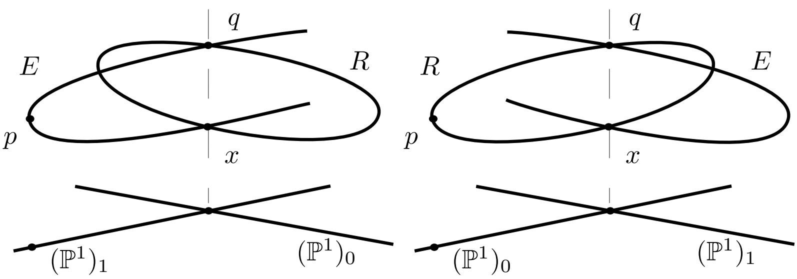
In the following, we will use the compactification $\overline{H}_{k,b}$ of $H_{k,b}$ by the space of admissible covers, introduced by Harris and Mumford in [**HM82**]. The map $\sigma$ extends to

$$\sigma\colon \overline{H}_{k,b} \to \overline{\mathcal{M}}_g.$$

In our case $g = 2k$, the image of this map is $\overline{\mathcal{M}}^1_{2k,k}$. It is classically known that the Hurwitz scheme is connected and its image in $\mathcal{M}_g$ (that is $\mathcal{M}^1_{2k,k}$ in our case) is irreducible (see for instance [**Ful69**]).

Similarly for a Schubert index $\alpha = (\alpha_0, \alpha_1)$ of type $1, k$ such that $\rho(g, 1, k, \alpha) = -1$ (and $\rho(g, 1, k) \geq 0$), the Hurwitz scheme $H_{k,b}(\alpha)$ (respectively $\overline{H}_{k,b}(\alpha)$) parameterizes $k$-sheeted (admissible) coverings $\pi\colon C \to \mathbb{P}^1$ with $b$ ordinary branch points $p_1, \ldots, p_b$ and one point $p$ with ramification profile described by $\alpha$ (see [**Dia85**, §5]). By forgetting the covering and keeping only the pointed source curve $(C, p)$, we obtain a map $\overline{H}_{k,b}(\alpha) \to \overline{\mathcal{M}}_{g,1}$ with image the pointed Brill-Noether divisor $\overline{\mathcal{M}}^1_{g,k}(\alpha)$.

Let us see these notions at work. Let $(C, p, q)$ be a two-pointed general curve of genus $g - 1 \geq 1$. In the following, we consider the curve $\overline{C}$ in $\overline{\mathcal{M}}_{g,1}$ obtained identifying the point $q$ with a moving point $x$ in $C$. In order to construct this family of curves, one blows up $C \times C$ at $(p, p)$ and $(q, q)$ and identifies the proper transforms $S_1$ and $S_2$ of the diagonal $\Delta_C$ and $q \times C$. This is a family $\pi\colon X \to C$ with a section corresponding to the proper transform of $p \times C$, hence there exists a map $C \to \overline{\mathcal{M}}_{g,1}$. We denote by $\overline{C}$ the image of $C$ in $\overline{\mathcal{M}}_{g,1}$.




(A) The case $2p \equiv q + x$ (B) The case $x = p$

FIGURE 2.4.1. The admissible covers for the two fibers of the family $\overline{C}$ when $g = 2$

LEMMA 2.4.1. *Let $g = 2$ and let $\mathcal{W}$ be the closure of the Weierstrass divisor in $\overline{\mathcal{M}}_{2,1}$. We have that*

$$\ell_{2,2} := \deg\left(\overline{C} \cdot \mathcal{W}\right) = 2.$$

PROOF. There are two points in $\overline{C}$ with an admissible cover of degree 2 with simple ramification at the marked point, and such admissible covers contribute with multiplicity one. Note that here $C$ is an elliptic curve. One admissible cover is for the fiber over $x$ such that $2p \equiv q + x$, and the other one for the fiber over $x = p$. In both cases the covering is determined by $|q + x|$ and there is a rational curve $R$ meeting $C$ in $q$ and $x$.

When $2p \equiv q + x$, the situation is as in [**HM82**, Thm. 6(a)]. Let $C' \to P$ be the corresponding admissible covering. If

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

is a general deformation of $[C' \to P]$ in $\overline{H}_{2,b}(0, 1)$, blowing down the curve $R$ we obtain a family of curves $\widetilde{\mathcal{C}} \to B$ with one ordinary double point. That is, $B$ meets $\Delta_0$ with multiplicity 2. Considering the involution of $[C' \to P]$ obtained interchanging the two ramification points of $R$, we see that the map $\overline{H}_{2,b}(0, 1) \to \overline{\mathcal{M}}_{2,1}$ is ramified at $[C' \to P]$. Hence $[C']$ is a transverse point of intersection of $\mathcal{W}$ with $\Delta_0$ and it follows that $\overline{C}$ and $\mathcal{W}$ meet transversally at $[C']$.

When $x = p$, the situation is similar. In a general deformation in $\overline{H}_{2,b}(0,1)$

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

of the corresponding admissible covering $[C' \to P]$, one sees that $C'$ is the only fiber of $\mathcal{C} \to B$ inside $\Delta_{00}$, and at each of the two nodes of $C'$, the space $\mathcal{C}$ has local equation $x \cdot y = t$. It follows that $C'$ is a transverse point of intersection of $\mathcal{W}$ with $\Delta_{00}$. Hence $C'$ is a transverse point of intersection of $\overline{C}$ with $\mathcal{W}$. See also [**Har84**, §3]. □

LEMMA 2.4.2. *Let $g = 2k-2 > 2$. The intersection of $\overline{C}$ with the pointed Brill-Noether divisor $\overline{\mathcal{M}}^1_{2k-2,k}(0,1)$ has degree*

$$\ell_{g,k} := \deg\left(\overline{C} \cdot \overline{\mathcal{M}}^1_{2k-2,k}(0,1)\right) = 2\frac{(2k-3)!}{(k-2)!(k-1)!}$$

*and is reduced.*

PROOF. Let us write the class of $\overline{\mathcal{M}}^1_{g,k}(0,1)$ as $a\lambda + c\psi - \sum b_i\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$. First we study the intersection of the curve $\overline{C}$ with the classes generating the Picard group. Let $\pi\colon \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ the map forgetting the marked point and $\sigma\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_{g,1}$ the section given by the marked point. Note that on $\overline{C}$ we have $\deg\psi = -\deg\pi_*(\sigma^2) = 1$, since the marked point is generically fixed and is blown-up in one fiber. Moreover $\deg\delta_{g-1} = 1$, since only one fiber contains a disconnecting node and the family is smooth at this point. The intersection with $\delta_0$ deserves more care. The family indeed is inside $\Delta_0$: the generic fiber has one non-disconnecting node and moreover the fiber over $x = p$ has two non-disconnecting nodes. We have to use [**HM98**, Lemma 3.94]. Then

$$\deg\delta_0 = \deg S_1^2 + \deg S_2^2 + 1 = -2(g-1) - 1 + 1 = 2 - 2g. \tag{2.4.1}$$

All other generating classes restrict to zero. Then

$$\deg\left(\overline{C} \cdot \left[\overline{\mathcal{M}}^1_{g,k}(0,1)\right]\right) = c + (2g-2)b_0 - b_{g-1}.$$

On the other hand, one has an explicit expression for the class of $\overline{\mathcal{M}}^1_{g,k}(0,1)$

$$\frac{(2k-4)!}{(k-2)!k!}\left(6(k+1)\lambda + 6(k-1)\psi - k\delta_0 + \sum_{i=1}^{g-1} 3(i+1)(2+i-2k)\delta_i\right)$$

(see [**Log03**, Thm. 4.5]), whence the first part of the statement.

Finally the intersection is reduced. Indeed, since the curve $C$ is general, an admissible cover with the desired property for a fiber of the family over $\overline{C}$ is determined by a unique linear series (see [**HM82**, pg. 75]). Moreover, reasoning as in the proof of the previous Lemma, one sees that $\overline{C}$ and $\overline{\mathcal{M}}^1_{g,k}(0,1)$ meet always transversally. $\square$

## 2.5. Limit linear series

The theory of limit linear series will be used. Let us quickly recall some notation and results. On a tree-like curve, a linear series or a limit linear series is called *generalized* if the line bundles involved are torsion-free (see [**EH87**, §1]). For a tree-like curve $C = Y_1 \cup \cdots \cup Y_s$ of arithmetic genus $g$ with disconnecting nodes at the points $\{p_{ij}\}_{ij}$, let $\{l_{Y_1}, \dots l_{Y_s}\}$ be a generalized limit linear series $\mathfrak{g}^r_d$ on $C$. Let $\{q_{ik}\}_k$ be smooth points on $Y_i$, $i = 1, \dots, s$. In [**EH86**] a moduli space of such limit series is constructed as a disjoint union of schemes on which the vanishing sequences of the aspects $l_{Y_i}$'s at the nodes are specified. A key property is the additivity of the adjusted Brill-Noether number, that is

$$\rho(g, r, d, \{\alpha^{l_{Y_i}}(q_{ik})\}_{ik}) \geq \sum_i \rho(Y_i, \{p_{ij}\}_j, \{q_{ik}\}_k).$$

The smoothing result [**EH86**, Cor. 3.7] assures the smoothability of dimensionally proper limit series. The following facts will ease the computations. The adjusted Brill-Noether number for any $\mathfrak{g}^r_d$ on one-pointed elliptic curves or on $n$-pointed rational curves is non-negative. For a general curve $C$ of arbitrary genus $g$, the adjusted Brill-Noether number for any $\mathfrak{g}^r_d$ with respect to $n$ general points is non-negative. Moreover, $\rho(C, y) \geq -1$ for any $y \in C$ and any $\mathfrak{g}^r_d$ (see [**EH89**]).

We will use the fact that if a curve of compact type has no limit linear series of type $\mathfrak{g}^r_d$, then it is not in the closure of the locus $\mathcal{M}^r_{g,d} \subset \mathcal{M}_g$ of smooth curves admitting a $\mathfrak{g}^r_d$.

## 2.6. Test surfaces

We are going to intersect both sides of (2.1.1) with several test surfaces. This will produce linear relations in the coefficients $A$.

The surfaces will be defined for arbitrary $g \geq 6$ (also odd values). Note that while the intersections of the surfaces with the generating classes (that is the left-hand sides of the relations we get) clearly depend solely on $g$, only the right-hand sides are specific to our problem of intersecting the test surfaces with $\overline{\mathcal{M}}^1_{2k,k}$.

When the base of a family is the product of two curves $C_1 \times C_2$, we will denote by $\pi_1$ and $\pi_2$ the obvious projections.

(S1) For $2 \leq i \leq \lfloor g/2 \rfloor$ consider the family of curves whose fibers are obtained identifying a moving point on a general curve $C_1$ of genus $i$ with a moving point on a general curve $C_2$ of genus $g - i$.

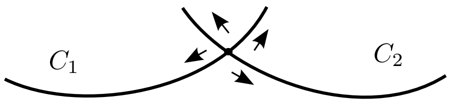



FIGURE 2.6.1. How the general fiber of a family in (S1) moves

The base of the family is the surface $C_1 \times C_2$. In order to construct this family, consider $C_1 \times C_1 \times C_2$ and $C_1 \times C_2 \times C_2$ and identify $\Delta_{C_1} \times C_2$ with $C_1 \times \Delta_{C_2}$. Let us denote this family by $X \to C_1 \times C_2$.

One has

$$\begin{aligned} \delta_i &= c_1\left(N_{(\Delta_{C_1}\times C_2)/X} \otimes N_{(C_1\times\Delta_{C_2})/X}\right) \\ &= -\pi_1^*(K_{C_1}) - \pi_2^*(K_{C_2}). \end{aligned}$$

Such surfaces are in the interior of the boundary of $\overline{\mathcal{M}}_g$. The only nonzero classes in codimension two are the ones considered in §2.2.

We claim that the intersection of these test surfaces with $\overline{\mathcal{M}}^1_{2k,k}$ has degree

$$T_i := \sum_{\substack{\alpha=(\alpha_0,\alpha_1) \\ \rho(i,1,k,\alpha)=-1}} n_{i,k,\alpha} \cdot n_{g-i,k,(k-1-\alpha_1,k-1-\alpha_0)}$$

(in the sum, $\alpha$ is a Schubert index of type $1, k$). Indeed by the remarks in §2.5, if $\{l_{C_1}, l_{C_2}\}$ is a limit linear series of type $\mathfrak{g}^1_k$ on the fiber over some $(x, y) \in C_1 \times C_2$, then the only possibility is $\rho(C_1, x) = \rho(C_2, y) = -1$. By §2.3.2, there are exactly $T_i$ points $(x, y)$ with this property, the linear series $l_{C_1}, l_{C_2}$ are uniquely determined and give an admissible cover of degree $k$. Thus to prove the claim we have to show that such points contribute with multiplicity one.

Let us first assume that $i > 2$. Let $\pi\colon C' \to P$ be one of these admissible covers of degree $k$, that is, $C'$ is stably equivalent to a certain fiber $C_1 \cup_{x\sim y} C_2$ of the family over $C_1 \times C_2$. Let us describe more precisely the admissible covering. Note that $P$ is the union of two rational curves $P = (\mathbb{P}^1)_1 \cup (\mathbb{P}^1)_2$. Moreover $\pi|_{C_1}\colon C_1 \to (\mathbb{P}^1)_1$ is the admissible covering of degree $k-\alpha_0$ defined by $l_{C_1}(-\alpha_0 x)$, $\pi|_{C_2}\colon C_2 \to (\mathbb{P}^1)_2$ is the admissible covering of degree $k - (k - 1 - \alpha_1) = \alpha_1 + 1$ defined by $l_{C_2}(-(k - 1 - \alpha_1)y)$, and $\pi$ has $\ell$-fold branching at $p := x \equiv y$ with $\ell := \alpha_1 + 1 - \alpha_0$. Finally there are $\alpha_0$ copies of $\mathbb{P}^1$ over $(\mathbb{P}^1)_1$ and further $k-1-\alpha_1$ copies over $(\mathbb{P}^1)_2$.

Such a cover has no automorphisms, hence the corresponding point $[\pi\colon C' \to P]$ in the Hurwitz scheme $\overline{H}_{k,b}$ is smooth, and moreover such a point is not fixed by any $\sigma \in \Sigma_b$. Let us embed $\pi\colon C' \to P$ in a one-dimensional family of admissible coverings

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

where locally near the point $p$

$$\begin{array}{lll} \mathcal{C} & \text{is} & r \cdot s = t, \\ \mathcal{P} & \text{is} & u \cdot v = t^{\ell}, \\ \pi & \text{is} & u = r^{\ell},\ v = s^{\ell} \end{array}$$

and $B := \operatorname{Spec} \mathbb{C}[[t]]$. Now $\mathcal{C}$ is a smooth surface and after contracting the extra curves $\mathbb{P}^1$, we obtain a family $\mathcal{C} \to B$ in $\overline{\mathcal{M}}_g$ transverse to $\Delta_i$ at the point $[C']$. Hence $(x, y)$ appears with multiplicity one in the intersection of $\overline{\mathcal{M}}^1_{2k,k}$ with $C_1 \times C_2$.

Finally if $i = 2$, then one has to take into account the automorphisms of the covers. To solve this, one has to work with the universal deformation space of the corresponding curve. The argument is similar (see [**HM82**, pg 80]).

For each $i$ we deduce the following relation

$$(2i-2)(2(g-i)-2)\left[2A_{\kappa_1^2} - A_{\omega^{(i)}} - A_{\omega^{(g-i)}}\right] = T_i.$$

Note that, if $i = g/2$, then $A_{\omega^{(i)}}$ and $A_{\omega^{(g-i)}}$ sum up.

(S2) Choose $i, j$ such that $2 \leq i \leq j \leq g-3$ and $i + j \leq g-1$. Take a general two-pointed curve $(F, p, q)$ of genus $g-i-j$ and attach at $p$ a moving point on a general curve $C_1$ of genus $i$ and at $q$ a moving point on a general curve $C_2$ of genus $j$.

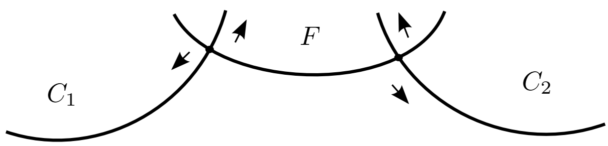



FIGURE 2.6.2. How the general fiber of a family in (S2) moves

The base of the family is $C_1 \times C_2$. To construct the family, consider $C_1 \times C_1 \times C_2$ and $C_1 \times C_2 \times C_2$ and identify $\Delta_{C_1} \times C_2$ and $C_1 \times \Delta_{C_2}$ with the general constant sections $p \times C_1 \times C_2$ and $q \times C_1 \times C_2$ of $F \times C_1 \times C_2 \to C_1 \times C_2$. Denote this family by $X \to C_1 \times C_2$. Then

$$\begin{aligned}
\delta_i &= c_1\left(N_{(\Delta_{C_1}\times C_2)/X} \otimes N_{(p\times C_1\times C_2)/X}\right)\\
&= -\pi_1^*(K_{C_1})\\
\delta_j &= c_1\left(N_{(C_1\times \Delta_{C_2})/X} \otimes N_{(q\times C_1\times C_2)/X}\right)\\
&= -\pi_2^*(K_{C_2})
\end{aligned}$$

and

$$\begin{aligned}
\delta_{ij} &= c_1\left(N_{(\Delta_{C_1}\times C_2)/X} \otimes N_{(p\times C_1\times C_2)/X}\right)\\
&\qquad \cdot c_1\left(N_{(C_1\times \Delta_{C_2})/X} \otimes N_{(q\times C_1\times C_2)/X}\right)\\
&= \pi_1^*(K_{C_1})\pi_2^*(K_{C_2}).
\end{aligned}$$

We claim that the intersection of these test surfaces with $\overline{\mathcal{M}}^1_{2k,k}$ has degree

$D_{ij} :=$

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \beta=(\beta_0,\beta_1)\\ \rho(i,1,k,\alpha)=-1\\ \rho(j,1,k,\beta)=-1}} n_{i,k,\alpha}\, n_{j,k,\beta}\, N_{g-i-j,k,(k-1-\alpha_1,k-1-\alpha_0),(k-1-\beta_1,k-1-\beta_0)}$$

(in the sum, $\alpha$ and $\beta$ are Schubert indices of type $1,k$.) Indeed by §2.5, if $\{l_{C_1}, l_F, l_{C_2}\}$ is a limit linear series of type $\mathfrak{g}^1_k$ on the fiber over some $(x,y) \in C_1 \times C_2$, then the only possibility is $\rho(C_1,x) = \rho(C_2,y) = -1$ while $\rho(F,p,q) = 0$. By §2.3.1 and §2.3.2, there are

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \beta=(\beta_0,\beta_1)\\ \rho(i,1,k,\alpha)=-1\\ \rho(j,1,k,\beta)=-1}} n_{i,k,\alpha}\, n_{j,k,\beta}$$

points $(x,y)$ in $C_1 \times C_2$ with this property, the $l_{C_1}, l_{C_2}$ are uniquely determined and there are

$$N := N_{g-i-j,k,(k-1-\alpha_1,k-1-\alpha_0),(k-1-\beta_1,k-1-\beta_0)}$$

choices for $l_F$. That is, there are $N$ points of $\overline{H}_{k,b}/\Sigma_b$ over $[C_1 \cup_{x\sim p} F \cup_{y\sim q} C_2] \in \overline{\mathcal{M}}^1_{2k,k}$ and $\overline{\mathcal{M}}^1_{2k,k}$ has $N$ branches at $[C_1 \cup_{x\sim p} F \cup_{y\sim q} C_2]$. The claim is thus equivalent to say that each branch meets $\Delta_{ij}$ transversely at $[C_1 \cup_{x\sim p} F \cup_{y\sim q} C_2]$.

The argument is similar to the previous case. Let $\pi\colon C' \to D$ be an admissible cover of degree $k$ with $C'$ stably equivalent to a certain fiber of the family over $C_1 \times C_2$. The image of a general deformation of $[C' \to D]$ in $\overline{H}_{k,b}$ to the universal deformation space of $C'$ meets $\Delta_{ij}$ only at $[C']$ and locally at the two nodes, the deformation space has equation $xy = t$. Hence $[C']$ is a transverse point of intersection of $\overline{\mathcal{M}}^1_{2k,k}$ with $\Delta_{ij}$ and the surface $C_1 \times C_2$ and $\overline{\mathcal{M}}^1_{2k,k}$ meet transversally.

For $i, j$ we obtain the following relation

$$(2i-2)(2j-2)\left[2A_{\kappa_1^2} + A_{\delta_{ij}}\right] = D_{ij}.$$

(S3) Let $(E, p, q)$ be a general two-pointed elliptic curve. Identify the point $q$ with a moving point $x$ on $E$ and identify the point $p$ with a moving point on a general curve $C$ of genus $g-2$.

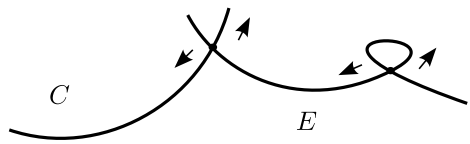



FIGURE 2.6.3. How the general fiber of a family in (S3) moves

The base of the family is $E \times C$. To construct the family, let us start from the blow-up $\widetilde{E \times E}$ of $E \times E$ at the points $(p,p)$ and $(q,q)$. Denote by $\sigma_p, \sigma_q, \sigma_\Delta$ the proper transforms respectively of $p \times E, q \times E, \Delta_E$. The family is the union of $\widetilde{E \times E} \times C$ and $E \times C \times C$ with $\sigma_q \times C$ identified with $\sigma_\Delta \times C$ and $\sigma_p \times C$ identified with $E \times \Delta_C$. We denote the family by $\pi\colon X \to E \times C$.

The study of the restriction of the generating classes in codimension one is similar to the case in the proof of Lemma 2.4.2. Namely

$$\begin{aligned}
\delta_0 &= -\pi_1^*(2q)\\
\delta_1 &= \pi_1^*(q)\\
\delta_{g-2} &= -\pi_1^*(p) - \pi_2^*(K_C).
\end{aligned}$$

Indeed the family is entirely contained inside $\Delta_0$: each fiber has a unique non-disconnecting node with the exception of the fibers over $p \times C$ which have two non-disconnecting nodes. Looking at the normalization of the family, fibers become smooth with the exception of the fibers over $p \times C$ which have now one non-disconnecting node, and the family is smooth at these points. It follows that $\delta_0 = \pi_*(\sigma_q \times C)^2 + \pi_*(\sigma_\Delta \times C)^2 + p \times C$. Only the fibers over $q \times C$ contain a node of type $\Delta_1$, and the family is smooth at these points. Finally the family is entirely inside $\Delta_{g-2}$ and $\delta_{g-2} = \pi_*(\sigma_p \times C)^2 + \pi_*(E \times \Delta_C)^2$. We note the following

$$\begin{aligned}\delta_{1,g-2} &= [\pi_1^*(q)][-\pi_2^*(K_C)]\\ \delta_{0,g-2} &= [-\pi_1^*(2q)][-\pi_2^*(K_C)].\end{aligned}$$

Let us study the intersection of this test surface with $\overline{\mathcal{M}}^1_{2k,k}$. Let $C' \to D$ be an admissible cover of degree $k$ with $C'$ stably equivalent to a certain fiber of the family. Clearly the only possibility is to map $E$ and $C$ to two different rational components of $D$ with $q$ and $x$ in the same fiber, and have a 2-fold ramification at $p$. From Lemma 2.4.1 there are two possibilities for the point $x \in E$, and there are $n_{g-2,k,(0,1)}$ points in $C$ where a degree $k$ covering has a 2-fold ramification. In each case the covering is unique up to isomorphism. The combination of the two makes

$$2n_{g-2,k,(0,1)}$$

admissible coverings. We claim that they count with multiplicity one.

The situation is similar to Lemma 2.4.1. The image of a general deformation of $[C' \to D]$ in $\overline{H}_{k,b}$ to the universal deformation space of $C'$ meets $\Delta_{00} \cap \Delta_2$ only at $[C']$. Locally at the three nodes, the deformation space has equation $xy = t$. Hence $[C']$ is a transverse point of intersection of $\overline{\mathcal{M}}^1_{2k,k}$ with $\Delta_{00} \cap \Delta_2$ and counts with multiplicity one in the intersection of the surface $E \times C$ with $\overline{\mathcal{M}}^1_{2k,k}$.

We deduce the following relation

$$(2(g-2)-2)\Big[4A_{\kappa_1^2} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}} - A_{\delta_{1,g-2}} + 2A_{\delta_{0,g-2}}\Big] = 2n_{g-2,k,(0,1)}.$$

(S4) For $2 \le i \le g-3$, let $(F, r, s)$ be a general two-pointed curve of genus $g-i-2$. Let $(E, p, q)$ be a general two-pointed elliptic curve and as above identify the point $q$ with a moving point $x$ on $E$. Finally identify the point $p \in E$ with $r \in F$ and identify the point $s \in F$ with a moving point on a general curve $C$ of genus $i$.

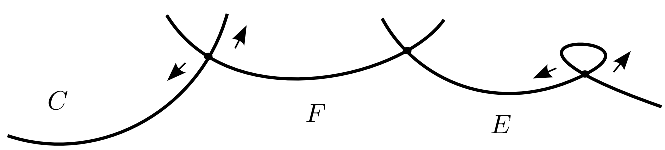



FIGURE 2.6.4. How the general fiber of a family in (S4) moves

The base of the family is $E \times C$. Let $\widetilde{E \times E}, \sigma_p, \sigma_q, \sigma_\Delta$ be as above. Then the family is the union of $\widetilde{E \times E} \times C, E \times C \times C$ and $F \times E \times C$ with the following identifications. First $\sigma_q \times C$ is identified with $\sigma_\Delta \times C$. Finally $\sigma_p \times E$ is identified with $r \times E \times C \subset F \times E \times C$, and $s \times E \times C \subset F \times E \times C$ with $E \times \Delta_C$.

The restriction of the generating classes in codimension one is

$$\begin{aligned}\delta_0 &= -\pi_1^*(2q)\\ \delta_1 &= \pi_1^*(q)\\ \delta_2 &= -\pi_1^*(p)\\ \delta_i &= -\pi_2^*(K_C)\end{aligned}$$

and one has the following restrictions

$$\begin{aligned}\delta_{1,i} &= [\pi_1^*(q)][-\pi_2^*(K_C)]\\ \delta_{0,i} &= [-\pi_1^*(2q)][-\pi_2^*(K_C)]\\ \delta_{2,i} &= [-\pi_1^*(p)][-\pi_2^*(K_C)].\end{aligned}$$

Suppose $C' \to D$ is an admissible covering of degree $k$ with $C'$ stably equivalent to a certain fiber of this family. The only possibility is to map $E, F, C$ to three different rational components of $D$, with a 2-fold ramification at $r$ and ramification prescribed by $\alpha = (\alpha_0, \alpha_1)$ at $s$, such that $\rho(i, 1, k, \alpha) = -1$. The condition on $\alpha$ is equivalent to

$$\rho(g-i-2, 1, k, (0,1), (k-1-\alpha_1, k-1-\alpha_0)) = 0.$$

Moreover, $q$ and $x$ have to be in the same fiber of such a covering. There are

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \rho(i,1,k,\alpha)=-1}} 2n_{i,k,\alpha}$$

fibers which admit an admissible covering with such properties (in the sum, $\alpha$ is a Schubert index of type $1, k$). While the restriction of the covering to $E$ and $C$ is uniquely determined up to isomorphism, there are

$$N := N_{g-i-2,k,(0,1),(k-1-\alpha_1,k-1-\alpha_0)}$$

choices for the restriction to $F$ up to isomorphism. As in (S2), this is equivalent to say that $\overline{\mathcal{M}}^1_{2k,k}$ has $N$ branches at $[C']$. Moreover, each branch meets the boundary transversally at $[C']$ (similarly to (S3)), hence $[C']$ counts with multiplicity one in the intersection of $E \times C$ with $\overline{\mathcal{M}}^1_{2k,k}$.

Finally, for each $i$ we deduce the following relation

$$\begin{aligned}(2i-2)&\Big[4A_{\kappa_1^2} - A_{\delta_{1,i}} + 2A_{\delta_{0,i}} + A_{\delta_{2,i}}\Big]\\ &= \sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \rho(i,1,k,\alpha)=-1}} 2N_{g-i-2,k,(0,1),(k-1-\alpha_1,k-1-\alpha_0)} \cdot n_{i,k,\alpha}.\end{aligned}$$

(S5) Identify a base point of a generic pencil of plane cubic curves with a moving point on a general curve $C$ of genus $g-1$.

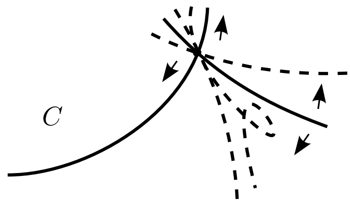



FIGURE 2.6.5. How the general fiber of a family in (S5) moves

The base of the family is $\mathbb{P}^1 \times C$. Let us construct this family. We start from an elliptic pencil $Y \to \mathbb{P}^1$ of degree 12 with zero section $\sigma$. To construct $Y$, blow up $\mathbb{P}^2$ in the nine points of intersection of two general cubics. Then consider $Y \times C$ and $\mathbb{P}^1 \times C \times C$ and identify $\sigma \times C$ with $\mathbb{P}^1 \times \Delta_C$. Let $x$ be the class of a point in $\mathbb{P}^1$. Then

$$\begin{array}{rcl} \lambda & = & \pi_1^*(x) \\ \delta_0 & = & 12\lambda \\ \delta_1 & = & -\pi_1^*(x) - \pi_2^*(K_C). \end{array}$$

Note that

$$\delta_{0,g-1} = [12\pi_1^*(x)][-\pi_2^*(K_C)].$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$. Indeed $C$ has no linear series with adjusted Brill-Noether number less than $-1$ at some point, and an elliptic curve or a rational nodal curve has no (generalized) linear series with adjusted Brill-Noether number less than 0 at some point. Adding, we see that no fiber of the family has a linear series with Brill-Noether number less than $-1$, hence

$$(2(g-1)-2)\left[2A_{\kappa_1^2} - 12A_{\delta_{0,g-1}} + 2A_{\delta_1^2} - A_{\lambda\delta_1}\right] = 0.$$

(S6) For $3 \le i \le g-3$ take a general curve $F$ of genus $i-1$ and attach at a general point $p$ an elliptic tail varying in a pencil of degree 12 and at another general point a moving point on a general curve $C$ of genus $g-i$.

The base of the family is $\mathbb{P}^1 \times C$. In order to construct the family, start from $Y \times C$ and $\mathbb{P}^1 \times C \times C$ and identify $\sigma \times C$ and $\mathbb{P}^1 \times \Delta_C$ with two general constant

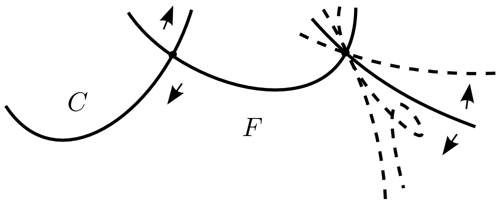



FIGURE 2.6.6. How the general fiber of a family in (S6) moves

sections of $F \times \mathbb{P}^1 \times C \to \mathbb{P}^1 \times C$. Here $Y, \sigma$ are as above. Then

$$\begin{aligned} \lambda &= \pi_1^*(x) \\ \delta_0 &= 12\lambda \\ \delta_1 &= -\pi_1^*(x) \\ \delta_{g-i} &= -\pi_2^*(K_C). \end{aligned}$$

Note that

$$\begin{aligned} \delta_{1,g-i} &= [-\pi_1^*(x)][-\pi_2^*(K_C)] \\ \delta_{0,g-i} &= [12\pi_1^*(x)][-\pi_2^*(K_C)]. \end{aligned}$$

Again $C$ has no linear series with adjusted Brill-Noether number less than $-1$ at some point, an elliptic curve or a rational nodal curve has no (generalized) linear series with adjusted Brill-Noether number less than 0 at some point and $F$ has no linear series with adjusted Brill-Noether number less than 0 at some general points. Adding, we see that no fiber of the family has a linear series with Brill-Noether number less than $-1$, hence

$$(2(g-i)-2)\left[2A_{\kappa_1^2} - A_{\lambda^{(i)}} + A_{\delta_{1,g-i}} - 12A_{\delta_{0,g-i}}\right] = 0.$$

In case $i = g-2$ we have

$$2\left[2A_{\kappa_1^2} - A_{\lambda\delta_2} + A_{\delta_{1,2}} - 12A_{\delta_{0,2}}\right] = 0.$$

(S7) Let $(E_1, p_1, q_1)$ and $(E_2, p_2, q_2)$ be two general pointed elliptic curves. Identify the point $q_i$ with a moving point $x_i$ in $E_i$, for $i = 1, 2$. Finally identify $p_1$ and $p_2$ with two general points $r_1, r_2$ on a general curve $F$ of genus $g-4$.

The base of the family is $E_1 \times E_2$. For $i = 1, 2$, let $\widetilde{E_i \times E_i}$ be the blow-up of $E_i \times E_i$ at $(p_i, p_i)$ and $(q_i, q_i)$. Denote by $\sigma_{p_i}, \sigma_{q_i}, \sigma_{\Delta_{E_i}}$ the proper transforms respectively of $p_i \times E_i, q_i \times E_i, \Delta_{E_i}$. The family is the union of $\widetilde{E_1 \times E_1} \times E_2$, $E_1 \times \widetilde{E_2 \times E_2}$ and $F \times E_1 \times E_2$ with the following identifications. First, $\sigma_{q_1} \times E_2$ and $E_1 \times \sigma_{q_2}$ are identified respectively with $\sigma_{\Delta_{E_1}} \times E_2$ and $E_1 \times \sigma_{\Delta_{E_2}}$. Then

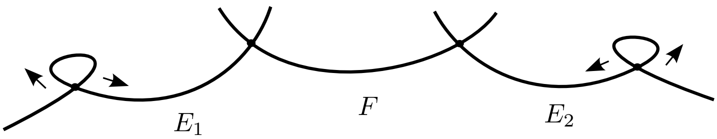



FIGURE 2.6.7. How the general fiber of a family in (S7) moves

$\sigma_{p_1} \times E_2$ and $E_1 \times \sigma_{p_2}$ are identified respectively with $r_1 \times E_1 \times E_2$ and $r_2 \times E_1 \times E_2$. We deduce

$$\begin{aligned} \delta_0 &= -\pi_1^*(2q_1) - \pi_2^*(2q_2) \\ \delta_1 &= \pi_1^*(q_1) + \pi_2^*(q_2) \\ \delta_2 &= -\pi_1^*(p_1) - \pi_2^*(p_2) \end{aligned}$$

and we note that

$$\begin{aligned} \delta_{2,2} &= \pi_1^*(p_1)\pi_2^*(p_2) \\ \delta_{1,2} &= -\pi_1^*(q_1)\pi_2^*(p_2) - \pi_2^*(q_2)\pi_1^*(p_1) \\ \delta_{1,1} &= \pi_1^*(q_1)\pi_2^*(q_2) \\ \delta_{00} &= \pi_1^*(2q_1)\pi_2^*(2q_2) \\ \delta_{02} &= \pi_1^*(2q_1)\pi_2^*(p_2) + \pi_2^*(2q_2)\pi_1^*(p_1) \\ \delta_{01} &= -\pi_1^*(q_1)\pi_2^*(2q_2) - \pi_2^*(q_2)\pi_1^*(2q_1). \end{aligned}$$

If a fiber of this family admits an admissible cover of degree $k$, then $r_1$ and $r_2$ have to be 2-fold ramification points, and $q_i$ and $x_i$ have to be in the same fiber, for $i = 1, 2$. From Lemma 2.4.1 there are only 4 fibers with this property, namely the fibers over $(p_1, p_2)$, $(p_1, \overline{q}_2)$, $(\overline{q}_1, p_2)$ and $(\overline{q}_1, \overline{q}_2)$, where $\overline{q}_i$ is such that $2p_i \equiv q_i + \overline{q}_i$ for $i = 1, 2$.

In these cases, the restriction of the covers to $E_1, E_2$ is uniquely determined up to isomorphism, while there are $N_{g-4,k,(0,1),(0,1)}$ choices for the restriction to $F$ up to isomorphism. As for (S3), such covers contribute with multiplicity one, hence we have the following relation

$$8A_{\kappa_1^2} + A_{\delta_{2,2}} - 2A_{\delta_{1,2}} + A_{\delta_{1,1}} + 2A_{\delta_1^2} + 8A_{\delta_0^2} + 4A_{\delta_{00}} + 4A_{\delta_{02}} - 4A_{\delta_{01}} \\ = 4N_{g-4,k,(0,1),(0,1)}.$$

(S8) Consider a general curve $F$ of genus $g-2$ and attach at two general points elliptic tails varying in pencils of degree 12.

The base of the family is $\mathbb{P}^1 \times \mathbb{P}^1$. Let us construct the family. Let $Y \to \mathbb{P}^1$ and $Y' \to \mathbb{P}^1$ be two elliptic pencils of degree 12, and let $\sigma$ and $\sigma'$ be the respective zero sections. Consider $Y \times \mathbb{P}^1$ and $\mathbb{P}^1 \times Y'$ and identify $\sigma \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \sigma'$ with

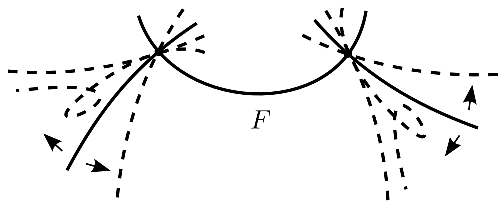



FIGURE 2.6.8. How the general fiber of a family in (S8) moves

two general constant sections of $F \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. If $x$ is the class of a point in $\mathbb{P}^1$, then

$$\begin{aligned} \lambda &= \pi_1^*(x) + \pi_2^*(x) \\ \delta_0 &= 12\lambda \\ \delta_1 &= -\lambda. \end{aligned}$$

Note that

$$\begin{aligned} \delta_{00} &= [12\pi_1^*(x)][12\pi_2^*(x)] \\ \delta_{1,1} &= [-\pi_1^*(x)][-\pi_2^*(x)] \\ \delta_{01} &= [12\pi_1^*(x)][-\pi_2^*(x)] + [-\pi_1^*(x)][12\pi_2^*(x)]. \end{aligned}$$

Studying the possibilities for the adjusted Brill-Noether numbers of the aspects of limit linear series on some fiber of this family, we see that this surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$2A_{\kappa_1^2} + 288A_{\delta_0^2} + 24A_{\lambda\delta_0} + 2A_{\delta_1^2} - 2A_{\lambda\delta_1} + 144A_{\delta_{00}} + A_{\delta_{1,1}} - 24A_{\delta_{01}} = 0.$$

(S9) For $2 \leq j \leq g-3$ let $R$ be a smooth rational curve, attach at the point $\infty \in R$ a general curve $F$ of genus $g-j-2$, attach at the points $0, 1 \in R$ two elliptic tails $E_1, E_2$ and identify a moving point in $R$ with a moving point on a general curve $C$ of genus $j$.

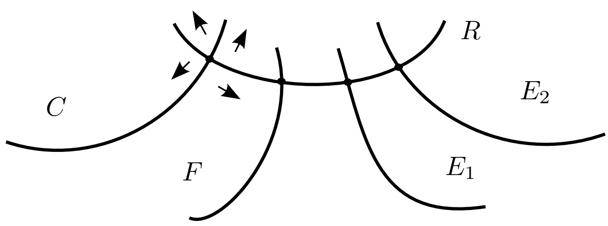



FIGURE 2.6.9. How the general fiber of a family in (S9) moves

The base of the family is $R \times C$. Let us start from a family $P \to R$ of four-pointed rational curves. Construct $P$ by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0,0),(1,1)$ and $(\infty,\infty)$, and consider the sections $\sigma_0,\sigma_1,\sigma_\infty$ and $\sigma_\Delta$ corresponding to the proper transforms of $0\times\mathbb{P}^1, 1\times\mathbb{P}^1, \infty\times\mathbb{P}^1$ and $\Delta_{\mathbb{P}^1}$.

To construct the family over $R\times C$, consider $P\times C$ and $R\times C\times C$. Identify $\sigma_\Delta\times C$ with $R\times\Delta_C$. Finally identify $\sigma_0\times C, \sigma_1\times C$ and $\sigma_\infty\times C$ respectively with general constant sections of the families $E_1\times R\times C, E_2\times R\times C$ and $F\times R\times C$. Then

$$\begin{aligned}
\delta_1 &= -\pi_1^*(0+1)\\
\delta_2 &= \pi_1^*(\infty)\\
\delta_j &= -\pi_1^*(K_{\mathbb{P}^1}+0+1+\infty)-\pi_2^*(K_C)\\
\delta_{g-j-2} &= -\pi_1^*(\infty)\\
\delta_{g-j-1} &= \pi_1^*(0+1).
\end{aligned}$$

If for some value of $j$ some of the above classes coincide (for instance, if $j=g-3$ then $\delta_1\equiv\delta_{g-j-2}$), then one has to sum up the contributions. Note that

$$\begin{aligned}
\delta_{1j} &= [-\pi_1^*(0+1)][-\pi_2^*(K_C)]\\
\delta_{j,g-j-2} &= [-\pi_1^*(\infty)][-\pi_2^*(K_C)]\\
\delta_{2,j} &= [\pi_1^*(\infty)][-\pi_2^*(K_C)]\\
\delta_{j,g-j-1} &= [\pi_1^*(0+1)][-\pi_2^*(K_C)].
\end{aligned}$$

As for (S8), this surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$(2j-2)\Big[2A_{\kappa_1^2}+2A_{\delta_{1j}}+A_{\delta_{j,g-j-2}}-A_{\delta_{2,j}}-2A_{\delta_{j,g-j-1}} - A_{\omega^{(j)}}-A_{\omega^{(g-j)}}\Big]=0.$$

Again, let us remark that for some value of $j$, some terms add up.

(S10) Let $(R_1,0,1,\infty)$ and $(R_2,0,1,\infty)$ be two three-pointed smooth rational curves, identify a moving point on $R_1$ with a moving point on $R_2$, attach a general pointed curve $F$ of genus $g-5$ to $\infty\in R_2$ and attach elliptic tails to all the other marked points.

The base of the family is $R_1\times R_2$. First construct two families of four-pointed rational curves $P_1\to R_1$ and $P_2\to R_2$ respectively with sections $\sigma_0,\sigma_1,\sigma_\infty,\sigma_\Delta$ and $\tau_0,\tau_1,\tau_\infty,\tau_\Delta$ as for the previous surface. Consider $P_1\times R_2$ and $R_1\times P_2$. Identify $\sigma_\Delta\times R_2$ with $R_1\times\tau_\Delta$. Finally identify $R_1\times\tau_\infty$ with a general constant section of $F\times R_1\times R_2$ and identify $\sigma_0\times R_2,\sigma_1\times R_2,\sigma_\infty\times R_2, R_1\times\tau_0, R_1\times\tau_1$ with the respective zero sections of five constant elliptic fibrations over $R_1\times R_2$.

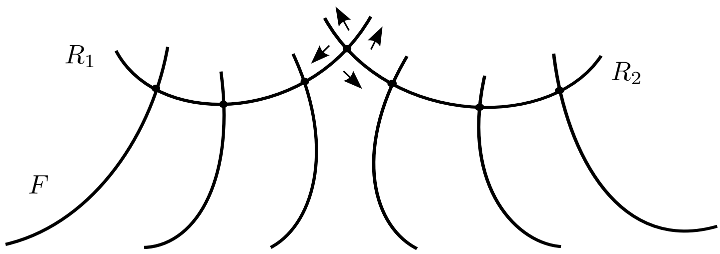



FIGURE 2.6.10. How the general fiber of a family in (S10) moves

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$. For $g > 8$

$$\begin{array}{rcl}
\delta_1 & = & -\pi_1^*(0+1+\infty) - \pi_2^*(0+1) \\
\delta_2 & = & \pi_1^*(0+1+\infty) + \pi_2^*(\infty) \\
\delta_3 & = & -\pi_1^*(K_{R_1}+0+1+\infty) - \pi_2^*(K_{R_2}+0+1+\infty) \\
\delta_{g-5} & = & -\pi_2^*(\infty) \\
\delta_{g-4} & = & \pi_2^*(0+1)
\end{array}$$

and note the restriction of the following classes

$$\begin{array}{rcl}
\delta_{1,1} & = & [-\pi_1^*(0+1+\infty)][-\pi_2^*(0+1)] \\
\delta_{1,g-5} & = & [-\pi_1^*(0+1+\infty)][-\pi_2^*(\infty)] \\
\delta_{1,3} & = & [-\pi_1^*(K_{R_1}+0+1+\infty)][-\pi_2^*(0+1)] \\
\delta_{3,g-5} & = & [-\pi_1^*(K_{R_1}+0+1+\infty)][-\pi_2^*(\infty)] \\
\delta_{1,g-3} & = & [-\pi_1^*(0+1+\infty)][-\pi_2^*(K_{R_2}+0+1+\infty)] \\
\delta_{2,g-3} & = & [\pi_1^*(0+1+\infty)][-\pi_2^*(K_{R_2}+0+1+\infty)] \\
\delta_{2,g-5} & = & [\pi_1^*(0+1+\infty)][-\pi_2^*(\infty)] \\
\delta_{1,2} & = & [\pi_1^*(0+1+\infty)][-\pi_2^*(0+1)] + [-\pi_1^*(0+1+\infty)][\pi_2^*(\infty)] \\
\delta_{1,g-4} & = & [-\pi_1^*(0+1+\infty)][\pi_2^*(0+1)] \\
\delta_{3,g-4} & = & [-\pi_1^*(K_{R_1}+0+1+\infty)][\pi_2^*(0+1)] \\
\delta_{2,3} & = & [-\pi_1^*(K_{R_1}+0+1+\infty)][\pi_2^*(\infty)] \\
\delta_{2,g-4} & = & [\pi_1^*(0+1+\infty)][\pi_2^*(0+1)] \\
\delta_{2,2} & = & [\pi_1^*(0+1+\infty)][\pi_2^*(\infty)].
\end{array}$$

It follows that

$$\begin{aligned}
2A_{\kappa_1^2} &+ 12A_{\delta_1^2} + 6A_{\delta_{1,1}} + 3A_{\delta_{1,g-5}} + 2A_{\delta_{1,3}} + A_{\delta_{3,g-5}} + 3A_{\delta_{1,g-3}} \\
&- A_{\omega^{(3)}} - A_{\omega^{(g-3)}} - 3(A_{\delta_{2,g-3}} + A_{\delta_{2,g-5}} + 2A_{\delta_{1,2}}) \\
&- 2(3A_{\delta_{1,g-4}} + A_{\delta_{3,g-4}}) - (3A_{\delta_{1,2}} + A_{\delta_{2,3}}) + 6A_{\delta_{2,g-4}} + 3A_{\delta_{2,2}} = 0.
\end{aligned}$$

For $g = 6$ the coefficient of $A_{\delta_1^2}$ is 18. When $g \in \{6, 7, 8\}$, note that some terms add up.

(S11) Consider a general curve $F$ of genus $g-4$, attach at a general point an elliptic tail varying in a pencil of degree 12 and identify a second general point with a moving point on a rational three-pointed curve $(R, 0, 1, \infty)$. Attach elliptic tails at the marked point on the rational curve.

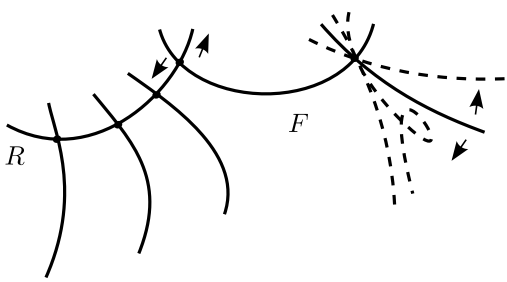



FIGURE 2.6.11. How the general fiber of a family in (S11) moves

The base of the family is $\mathbb{P}^1 \times R$. Consider the elliptic fibration $Y$ over $\mathbb{P}^1$ with zero section $\sigma$ as in (S5), and the family $P$ over $R$ with sections $\sigma_0, \sigma_1, \sigma_\infty, \sigma_\Delta$ as in (S9). Identify $\sigma \times R \subset Y \times R$ and $\mathbb{P}^1 \times \sigma_\Delta \subset \mathbb{P}^1 \times P$ with two general constant sections of $F \times \mathbb{P}^1 \times R$. Finally identify $\mathbb{P}^1 \times \sigma_0, \mathbb{P}^1 \times \sigma_1, \mathbb{P}^1 \times \sigma_\infty \subset \mathbb{P}^1 \times P$ with the respective zero sections of three constant elliptic fibrations over $\mathbb{P}^1 \times R$. Then

$$\begin{aligned}
\lambda &= \pi_1^*(x) \\
\delta_0 &= 12\lambda \\
\delta_1 &= -\pi_1^*(x) - \pi_2^*(0 + 1 + \infty) \\
\delta_2 &= \pi_2^*(0 + 1 + \infty) \\
\delta_3 &= -\pi_2^*(K_R + 0 + 1 + \infty).
\end{aligned}$$

Note the restriction of the following classes

$$\begin{aligned}
\delta_{1,1} &= [-\pi_1^*(x)][-\pi_2^*(0+1+\infty)]\\
\delta_{1,3} &= [-\pi_1^*(x)][-\pi_2^*(K_R+0+1+\infty)]\\
\delta_{01} &= [12\pi_1^*(x)][-\pi_2^*(0+1+\infty)]\\
\delta_{03} &= [12\pi_1^*(x)][-\pi_2^*(K_R+0+1+\infty)]\\
\delta_{02} &= [12\pi_1^*(x)][\pi_2^*(0+1+\infty)]\\
\delta_{1,2} &= [-\pi_1^*(x)][\pi_2^*(0+1+\infty)].
\end{aligned}$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$2A_{\kappa_1^2} - A_{\lambda^{(g-3)}} + 6A_{\delta_1^2} + 3A_{\delta_{1,1}} - 3A_{\lambda\delta_1} + A_{\delta_{1,3}} - 36A_{\delta_{01}} - 12A_{\delta_{03}} + 3\left[A_{\lambda\delta_2} + 12A_{\delta_{02}} - A_{\delta_{1,2}}\right] = 0.$$

(S12) Let $R$ be a rational curve, attach at the points 0 and 1 two fixed elliptic tails, attach at the point $\infty$ an elliptic tail moving in a pencil of degree 12 and identify a moving point in $R$ with a general point on a general curve $F$ of genus $g-3$.

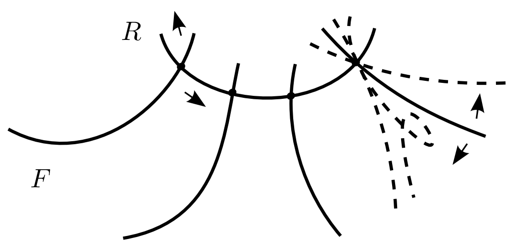



FIGURE 2.6.12. How the general fiber of a family in (S12) moves

The base of the family is $\mathbb{P}^1 \times R$. Let $Y, \sigma$ and $P, \sigma_0, \sigma_1, \sigma_\infty, \sigma_\Delta$ be as above. Identify $\sigma \times R \subset Y \times R$ with $\mathbb{P}^1 \times \sigma_\infty \subset \mathbb{P}^1 \times P$, and $\mathbb{P}^1 \times \sigma_\Delta \subset \mathbb{P}^1 \times P$ with a general constant section of $F \times \mathbb{P}^1 \times R$. Finally identify $\mathbb{P}^1 \times \sigma_0, \mathbb{P}^1 \times \sigma_1$ with the zero sections of two constant elliptic fibrations over $\mathbb{P}^1 \times R$. Then

$$\begin{aligned}
\lambda &= \pi_1^*(x)\\
\delta_0 &= 12\lambda\\
\delta_1 &= -\pi_1^*(x) - \pi_2^*(\infty+0+1)\\
\delta_2 &= \pi_2^*(\infty+0+1)\\
\delta_3 &= -\pi_2^*(K_{\mathbb{P}^1}+0+1+\infty).
\end{aligned}$$

Let us note the following restrictions

$$\begin{aligned}
\delta_{01} &= [12\pi_1^*(x)][-\pi_2^*(0+1)]\\
\delta_{0,g-3} &= [12\pi_1^*(x)][-\pi_2^*(K_{\mathbb{P}^1}+0+1+\infty)]\\
\delta_{0,g-1} &= [12\pi_1^*(x)][-\pi_2^*(\infty)]\\
\delta_{1,1} &= [-\pi_1^*(x)][-\pi_2^*(0+1)]\\
\delta_{1,g-3} &= [-\pi_1^*(x)][-\pi_2^*(K_{\mathbb{P}^1}+0+1+\infty)]\\
\delta_{0,g-2} &= [12\pi_1^*(x)][\pi_2^*(0+1)]\\
\delta_{1,g-2} &= [-\pi_1^*(x)][\pi_2^*(0+1)]\\
\delta_{02} &= [12\pi_1^*(x)][\pi_2^*(\infty)]\\
\delta_{1,2} &= [-\pi_1^*(x)][\pi_2^*(\infty)].
\end{aligned}$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$\begin{aligned}
2A_{\kappa_1^2} - 3A_{\lambda\delta_1} - 24A_{\delta_{01}} - 12A_{\delta_{0,g-3}} - 12A_{\delta_{0,g-1}} + 6A_{\delta_1^2} + 2A_{\delta_{1,1}} + A_{\delta_{1,g-3}}\\
- A_{\lambda^{(3)}} + 2(A_{\lambda\delta_2} + 12A_{\delta_{0,g-2}} - A_{\delta_{1,g-2}}) + (A_{\lambda\delta_2} + 12A_{\delta_{02}} - A_{\delta_{1,2}}) = 0.
\end{aligned}$$

(S13) Let $(C,p,q)$ be a general two-pointed curve of genus $g-3$ and identify the point $q$ with a moving point $x$ on $C$. Let $(E,r,s)$ be a general two-pointed elliptic curve and identify the point $s$ with a moving point $y$ on $E$. Finally identify the points $p$ and $r$.

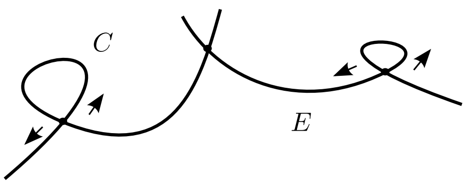



FIGURE 2.6.13. How the general fiber of a family in (S13) moves

The base of the family is $C \times E$. Let $\widetilde{C \times C}$ (respectively $\widetilde{E \times E}$) be the blow-up of $C \times C$ at $(p,p)$ and $(q,q)$ (respectively of $E \times E$ at $(r,r)$ and $(s,s)$). Let $\tau_p, \tau_q, \tau_\Delta$ (respectively $\sigma_r, \sigma_s, \sigma_\Delta$) be the proper transform of $p \times C, q \times C, \Delta_C$ (respectively $r \times E, s \times E, \Delta_E$) and identify $\tau_q$ with $\tau_\Delta$ (respectively $\sigma_s$ with $\sigma_\Delta$). Finally identify $\tau_p \times E$ with $C \times \sigma_r$. Then from the proof of Lemma 2.4.2, we have

$$\begin{aligned}
\delta_0 &= -\pi_1^*(K_C+2q) - \pi_2^*(2s)\\
\delta_1 &= \pi_1^*(q) + \pi_2^*(s)\\
\delta_2 &= -\pi_1^*(p) - \pi_2^*(r)
\end{aligned}$$

and note that

$$\begin{aligned}
\delta_{00} &= [-\pi_1^*(K_C+2q)][-\pi_2^*(2s)] \\
\delta_{02} &= [-\pi_1^*(K_C+2q)][-\pi_2^*(r)] \\
\delta_{0,g-2} &= [-\pi_2^*(2s)][-\pi_1^*(p)] \\
\delta_{01} &= [-\pi_1^*(K_C+2q)][\pi_2^*(s)] + [-\pi_2^*(2s)][\pi_1^*(q)] \\
\delta_{1,g-2} &= [-\pi_1^*(p)][\pi_2^*(s)] \\
\delta_{1,2} &= [\pi_1^*(q)][-\pi_2^*(r)] \\
\delta_{1,1} &= [\pi_1^*(q)][\pi_2^*(s)].
\end{aligned}$$

If a fiber of this family admits an admissible covering of degree $k$, then such a covering has a 2-fold ramification at the point $p \sim r$, $q$ is in the same fiber as $x$, and $s$ is in the same fiber as $y$. By Lemma 2.4.1 and Lemma 2.4.2 there are 2 points in $E$ and $\ell_{g-2,k}$ points in $C$ with such a property, and the cover is unique up to isomorphism. Reasoning as in (S3), one shows that each cover contributes with multiplicity one. It follows that

$$\begin{aligned}
&2(g-3)\left[4A_{\kappa_1^2} + 2A_{\delta_{00}} + 4A_{\delta_0^2} + A_{\delta_{02}}\right] + 2A_{\delta_{0,g-2}} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}} \\
&\quad - \left[2(g-3)A_{\delta_{01}} + A_{\delta_{1,g-2}}\right] - \left[2A_{\delta_{01}} + A_{\delta_{1,2}}\right] + \left[A_{\delta_{1,1}} + 2A_{\delta_1^2}\right] = 2\cdot \ell_{g-2,k}.
\end{aligned}$$

(S14) Let $(C,p,q)$ be a general two-pointed curve of genus $g-2$, attach at $p$ an elliptic tail moving in a pencil of degree 12 and identify $q$ with a moving point on $C$.

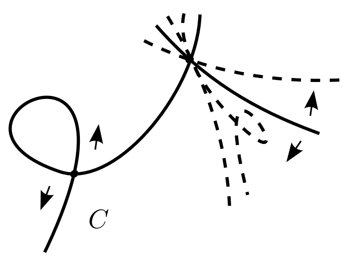



FIGURE 2.6.14. How the general fiber of a family in (S14) moves

The base of this family is $C\times\mathbb{P}^1$. Let $\widetilde{C\times C}$ be the blow-up of $C\times C$ at the points $(p,p)$ and $(q,q)$. Let $\tau_p, \tau_q, \tau_\Delta$ be the proper transform of $p\times C, q\times C, \Delta$ and identify $\tau_q$ with $\tau_\Delta$. Then consider $Y, \sigma$ as in (S5) and identify $C\times\sigma$ with

$\tau_p \times \mathbb{P}^1$. Then

$$\begin{aligned}\lambda &= \pi_2^*(x)\\ \delta_0 &= 12\lambda - \pi_1^*(K_C + 2q)\\ \delta_1 &= \pi_1^*(q) - \pi_1^*(p) - \lambda.\end{aligned}$$

Note that

$$\begin{aligned}\delta_{00} &= [12\pi_2^*(x)][-\pi_1^*(K_C + 2q)]\\ \delta_{01} &= [\pi_1^*(q)][12\pi_2^*(x)] + [-\pi_1^*(K_C + 2q)][-\pi_2^*(x)]\\ \delta_{0,g-1} &= [-\pi_1^*(p)][12\pi_2^*(x)]\\ \delta_{1,1} &= [\pi_1^*(q)][-\pi_2^*(x)].\end{aligned}$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$(2g-4)\left[2A_{\kappa_1^2} - A_{\lambda\delta_0} - 24A_{\delta_0^2} - 12A_{\delta_{00}} + A_{\delta_{01}}\right] \\ - 12A_{\delta_{0,g-1}} + (12A_{\delta_{01}} - A_{\delta_{1,1}}) = 0.$$

(S15) Let $C$ be a general curve of genus $g-1$ and consider the surface $C \times C$ with fiber $C/(p \sim q)$ over $(p, q)$.

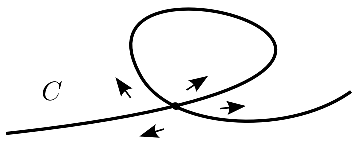



FIGURE 2.6.15. How the general fiber of a family in (S15) moves

To construct the family, start from $p_{2,3} \colon C \times C \times C \to C \times C$, blow up the diagonal $\Delta \subset C \times C \times C$ and then identify the proper transform of $\Delta_{1,2} := p_{1,2}^*(\Delta)$ with the proper transform $\Delta_{1,3} := p_{1,3}^*(\Delta)$. Then

$$\begin{aligned}\delta_0 &= -(\pi_1^* K_C + \pi_2^* K_C + 2\Delta)\\ \delta_1 &= \Delta.\end{aligned}$$

The class $\kappa_2$ has been computed in [**Fab90a**, §2.1 (1)]. The curve $C$ has no generalized linear series with Brill-Noether number less than 0, hence

$$(8g^2 - 26g + 20)A_{\kappa_1^2} + (2g-4)A_{\kappa_2} + (4-2g)A_{\delta_1^2} + 8(g-1)(g-2)A_{\delta_0^2} = 0.$$

(S16) For $\lfloor g/2 \rfloor \leq i \leq g-2$, take a general curve $C$ of genus $i$ and attach an elliptic curve $E$ and a general pointed curve $F$ of genus $g-i-1$ at two varying points in $C$.

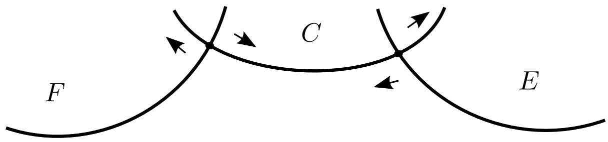



FIGURE 2.6.16. How the general fiber of a family in (S16) moves

To construct the family, blow up the diagonal $\Delta$ in $C \times C \times C$ as before, and then identify the proper transform of $\Delta_{1,2}$ with the zero section of a constant elliptic fibration over $C \times C$, and identify the proper transform $\Delta_{1,3}$ with a general constant section of $F \times C \times C$. For $i < g-2$

$$\begin{aligned} \delta_1 &= -\pi_1^* K_C - \Delta \\ \delta_{g-i-1} &= -\pi_2^* K_C - \Delta \\ \delta_i &= \Delta \end{aligned}$$

while for $i = g-2$ the $\delta_1$ is the sum of the above $\delta_1$ and $\delta_{g-i-1}$.

Note that replacing the tail of genus $g-i-1$ with an elliptic tail does not affect the computation of the class $\kappa_2$, hence we can use the count from [**Fab90b**, §3 ($\gamma$)], that is $\kappa_2 = 2i-2$. About the $\omega$ classes, on these test surfaces one has $\omega^{(i)} = -\delta_i^2$ and $\omega^{(i+1)} = -\delta_{i+1}^2 = -\delta_{g-i-1}^2$. Finally note that $\delta_{1,g-i-1}$ is the product of the $c_1$'s coming from the two nodes, that is, $\delta_{1,g-i-1} = \delta_1 \delta_{g-i-1}$.

If a fiber of this family has a $\mathfrak{g}^1_k$ limit linear series $\{l_E, l_C, l_F\}$, then necessarily the adjusted Brill-Noether number has to be zero on $F$ and $E$, and $-2$ on $C$. Note that in any case $l_E = |2 \cdot 0_E|$. From §2.3.3 there are

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1) \\ \rho(i,1,k,\alpha)=-1}} m_{i,k,\alpha}$$

pairs in $C$ with such a property, $l_C$ is also uniquely determined and there are $N_{g-i-1,d,(d-1-\alpha_1,d-1-\alpha_0)}$ choices for $l_F$. With a similar argument to (S2), such pairs contribute with multiplicity one.

All in all for $i < g-2$

$$\begin{aligned} (2i-2)\big[(4i-1)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(i)}} - A_{\omega^{(i+1)}} + A_{\delta_1^2} + (2i-1)A_{\delta_{1,g-i-1}}\big] \\ = \sum_{\substack{0 \leq \alpha_0 \leq \alpha_1 \leq k-1 \\ \alpha_0+\alpha_1=g-i-1}} m_{i,k,(\alpha_0,\alpha_1)} \cdot N_{g-i-1,k,(k-1-\alpha_1,k-1-\alpha_0)} \end{aligned}$$

while for $i = g-2$

$$(2g-6)\big[(4g-9)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(g-2)}} + (4g-8)A_{\delta_1^2} + (2g-5)A_{\delta_{1,1}}\big] = m_{g-2,k,(0,1)}.$$

(S17) Consider a general element in $\theta_1$, vary the elliptic curve in a pencil of degree 12 and vary one point on the elliptic curve.

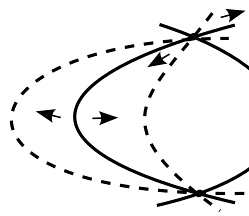


FIGURE 2.6.17. How the general fiber of a family in (S17) moves

The base of this family is the blow up of $\mathbb{P}^2$ in the nine points of intersection of two general cubic curves. Let us denote by $H$ the pull-back of an hyperplane section in $\mathbb{P}^2$, by $\Sigma$ the sum of the nine exceptional divisors and by $E_0$ one of them. We have

$$\begin{aligned} \lambda &= 3H - \Sigma \\ \delta_0 &= 30H - 10\Sigma - 2E_0 \\ \delta_1 &= E_0 \end{aligned}$$

(see also [**Fab89**, §2 (9)]). Replacing the component of genus $g-2$ with a curve of genus 2, we obtain a surface in $\overline{\mathcal{M}}_4$. The computation of the class $\kappa_2$ remains unaltered, that is $\kappa_2 = 1$ (see [**Fab90b**, §3 ($\iota$)]). Similarly for $\delta_{00}$ and $\theta_1$, while $\delta_{0,g-1}$ correspond to the value of $\delta_{01a}$ on the surface in $\overline{\mathcal{M}}_4$.

Let us study the intersection with $\overline{\mathcal{M}}^1_{2k,k}$. An admissible cover for some fiber of this family would necessarily have the two nodes in the same fiber, which is impossible, since the two points are general on the component of genus $g-2$. We deduce the following relation

$$3A_{\kappa_1^2} + A_{\kappa_2} - 2A_{\lambda\delta_0} + A_{\lambda\delta_1} - 44A_{\delta_0^2} - A_{\delta_1^2} + 12A_{\delta_{0,g-1}} - 12A_{\delta_{00}} + A_{\theta_1} = 0.$$

(S18) For $2 \le i \le \lfloor (g+1)/2 \rfloor$ we consider a general curve of type $\delta_{i-1,g-i}$ and we vary the central elliptic curve $E$ in a pencil of degree 12 and one of the points on $E$.

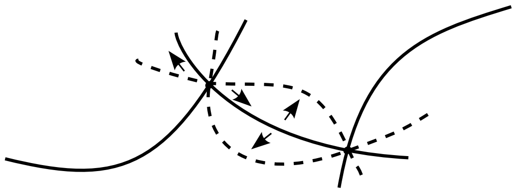


FIGURE 2.6.18. How the general fiber of a family in (S18) moves

The base of this family is the same surface as in (S17). For $i \geq 3$ we have

$$\begin{aligned} \lambda &= 3H - \Sigma \\ \delta_0 &= 12\lambda \\ \delta_1 &= E_0 \\ \delta_{i-1} &= -3H + \Sigma - E_0 \\ \delta_{g-i} &= -3H + \Sigma - E_0 \end{aligned}$$

while for $i = 2$ the $\delta_1$ is the sum of the above $\delta_1$ and $\delta_{i-1}$, that is $\delta_1 = -3H + \Sigma$ (see also [**Fab90b**, §3 ($\lambda$)]).

Note that replacing the two tails of genus $i-1$ and $g-i$ with tails of genus 1 and 2, we obtain a surface in $\overline{\mathcal{M}}_4$. The computation of the class $\kappa_2$ remains unaltered, that is $\kappa_2 = 1$ (see [**Fab90b**, §3 ($\lambda$)]). Moreover, on these test surfaces $\omega^{(i)} = -\delta_i^2 = -\delta_{g-i}^2$ and for $i \geq 3$, $\omega^{(g-i+1)} = -\delta_{g-i+1}^2 = -\delta_{i-1}^2$ hold, while $\lambda^{(i)} = \lambda\delta_i = \lambda\delta_{g-i}$ for $i \geq 3$ and $\lambda^{(g-i+1)} = \lambda\delta_{g-i+1} = \lambda\delta_{i-1}$ for $i \geq 4$. All fibers are in $\delta_{i-1,g-i}$, hence $\delta_{i-1,g-i}$ is the product of the $c_1$'s of the two nodes, that is, $\delta_{i-1,g-i} = \delta_{i-1} \cdot \delta_{g-i}$. Note that on these surfaces, $\delta_{0,i-1} = \delta_0\delta_{i-1}$ and $\delta_{0,g-i} = \delta_0\delta_{g-i}$. There are exactly 12 fibers which contribute to $\theta_{i-1}$, namely when the elliptic curve degenerates into a rational nodal curve and the moving point hits the non-disconnecting node. Similarly, there are 12 fibers which contribute to $\delta_{0,g-1}$, namely when the elliptic curve degenerates into a rational nodal curve and the moving point hits the disconnecting node.

These surfaces are disjoint from $\overline{\mathcal{M}}^1_{2k,k}$. Indeed the two tails of genus $i-1$ and $g-i$ have no linear series with adjusted Brill-Noether number less than 0 at general points. Moreover an elliptic curve has no $\mathfrak{g}^1_k$ with adjusted Brill-Noether number less than $-1$ at two arbitrary points. Finally a rational nodal curve has no generalized linear series with adjusted Brill-Noether number less than 0 at arbitrary points.

It follows that for $i \geq 4$ we have

$$\begin{aligned} 3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(i)}} - A_{\omega^{(g-i+1)}} - A_{\delta_1^2} + A_{\delta_{i-1,g-i}} - A_{\lambda^{(i)}} - A_{\lambda^{(g-i+1)}} \\ + A_{\lambda\delta_1} - 12A_{\delta_{0,i-1}} - 12A_{\delta_{0,g-i}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_{i-1}} = 0, \end{aligned}$$

when $i = 3$

$$3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(3)}} - A_{\omega^{(g-2)}} - A_{\delta_1^2} + A_{\delta_{2,g-3}} - A_{\lambda^{(3)}} - A_{\lambda\delta_2} \\ + A_{\lambda\delta_1} - 12A_{\delta_{0,2}} - 12A_{\delta_{0,g-3}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_2} = 0,$$

and when $i = 2$

$$3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(2)}} + A_{\delta_{1,g-2}} \\ - A_{\lambda\delta_2} - 12A_{\delta_{0,1}} - 12A_{\delta_{0,g-2}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_1} = 0.$$

## 2.7. The result

In (S1)-(S18) we have constructed

$$\lfloor (g^2-1)/4 \rfloor + 3g - 1$$

linear relations in the coefficients $A$. Let us collect here all the relations. For $2 \le i \le \lfloor g/2 \rfloor$ from (S1) we obtain

$$2A_{\kappa_1^2} - A_{\omega^{(i)}} - A_{\omega^{(g-i)}} = \frac{T_i}{(2i-2)(2(g-i)-2)},$$

from (S2) for $2 \le i \le j \le g-3$ and $i + j \le g - 1$

$$2A_{\kappa_1^2} + A_{\delta_{ij}} = \frac{D_{ij}}{(2i-2)(2j-2)},$$

from (S3)

$$4A_{\kappa_1^2} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}} - A_{\delta_{1,g-2}} + 2A_{\delta_{0,g-2}} = \frac{n_{g-2,k,(0,1)}}{g-3},$$

from (S4) for $2 \le i \le g-3$

$$4A_{\kappa_1^2} - A_{\delta_{1,i}} + 2A_{\delta_{0,i}} + A_{\delta_{2,i}} = \frac{D_{2,i}}{6(i-1)},$$

from (S5)

$$2A_{\kappa_1^2} - 12A_{\delta_{0,g-1}} + 2A_{\delta_1^2} - A_{\lambda\delta_1} = 0,$$

from (S6) for $3 \le i \le g-3$

$$2A_{\kappa_1^2} - A_{\lambda^{(i)}} + A_{\delta_{1,g-i}} - 12A_{\delta_{0,g-i}} = 0$$

and

$$2A_{\kappa_1^2} - A_{\lambda\delta_2} + A_{\delta_{1,2}} - 12A_{\delta_{0,2}} = 0,$$

from (S7)

$$8A_{\kappa_1^2} + A_{\delta_{2,2}} - 2A_{\delta_{1,2}} + A_{\delta_{1,1}} + 2A_{\delta_1^2} + 8A_{\delta_0^2} + 4A_{\delta_{00}} + 4A_{\delta_{02}} - 4A_{\delta_{01}} \\ = 4N_{g-4,k,(0,1),(0,1)},$$

from (S8)

$$2A_{\kappa_1^2}+288A_{\delta_0^2}+24A_{\lambda\delta_0}+2A_{\delta_1^2}-2A_{\lambda\delta_1}+144A_{\delta_{00}}+A_{\delta_{1,1}}-24A_{\delta_{01}}=0,$$

from (S9) for $2\le j\le g-3$

$$2A_{\kappa_1^2}+2A_{\delta_{1j}}+A_{\delta_{j,g-j-2}}-A_{\delta_{2,j}}-2A_{\delta_{j,g-j-1}}-A_{\omega^{(j)}}-A_{\omega^{(g-j)}}=0,$$

from (S10) for $g>6$

$$\begin{aligned}2A_{\kappa_1^2}+12A_{\delta_1^2}+6A_{\delta_{1,1}}+3A_{\delta_{1,g-5}}+2A_{\delta_{1,3}}+A_{\delta_{3,g-5}}+3A_{\delta_{1,g-3}}\\ -A_{\omega^{(3)}}-A_{\omega^{(g-3)}}-3(A_{\delta_{2,g-3}}+A_{\delta_{2,g-5}}+2A_{\delta_{1,2}})\\ -2(3A_{\delta_{1,g-4}}+A_{\delta_{3,g-4}})-(3A_{\delta_{1,2}}+A_{\delta_{2,3}})+6A_{\delta_{2,g-4}}+3A_{\delta_{2,2}}=0\end{aligned}$$

while for $g=6$

$$\begin{aligned}2A_{\kappa_1^2}+18A_{\delta_1^2}+6A_{\delta_{1,1}}+3A_{\delta_{1,g-5}}+2A_{\delta_{1,3}}+A_{\delta_{3,g-5}}+3A_{\delta_{1,g-3}}\\ -A_{\omega^{(3)}}-A_{\omega^{(g-3)}}-3(A_{\delta_{2,g-3}}+A_{\delta_{2,g-5}}+2A_{\delta_{1,2}})\\ -2(3A_{\delta_{1,g-4}}+A_{\delta_{3,g-4}})-(3A_{\delta_{1,2}}+A_{\delta_{2,3}})+6A_{\delta_{2,g-4}}+3A_{\delta_{2,2}}=0,\end{aligned}$$

from (S11)

$$\begin{aligned}2A_{\kappa_1^2}-A_{\lambda^{(g-3)}}+6A_{\delta_1^2}+3A_{\delta_{1,1}}-3A_{\lambda\delta_1}+A_{\delta_{1,3}}-36A_{\delta_{01}}-12A_{\delta_{03}}\\ +3\left[A_{\lambda\delta_2}+12A_{\delta_{02}}-A_{\delta_{1,2}}\right]=0,\end{aligned}$$

from (S12)

$$\begin{aligned}2A_{\kappa_1^2}-3A_{\lambda\delta_1}-24A_{\delta_{01}}-12A_{\delta_{0,g-3}}-12A_{\delta_{0,g-1}}+6A_{\delta_1^2}+2A_{\delta_{1,1}}+A_{\delta_{1,g-3}}\\ -A_{\lambda^{(3)}}+2(A_{\lambda\delta_2}+12A_{\delta_{0,g-2}}-A_{\delta_{1,g-2}})+(A_{\lambda\delta_2}+12A_{\delta_{02}}-A_{\delta_{1,2}})=0,\end{aligned}$$

from (S13)

$$\begin{aligned}2(g-3)\left[4A_{\kappa_1^2}+2A_{\delta_{00}}+4A_{\delta_0^2}+A_{\delta_{02}}\right]+2A_{\delta_{0,g-2}}-A_{\omega^{(2)}}-A_{\omega^{(g-2)}}\\ -\left[2(g-3)A_{\delta_{01}}+A_{\delta_{1,g-2}}\right]-\left[2A_{\delta_{01}}+A_{\delta_{1,2}}\right]+\left[A_{\delta_{1,1}}+2A_{\delta_1^2}\right]=2\cdot\ell_{g-2,k},\end{aligned}$$

from (S14)

$$\begin{aligned}(2g-4)\left[2A_{\kappa_1^2}-A_{\lambda\delta_0}-24A_{\delta_0^2}-12A_{\delta_{00}}+A_{\delta_{01}}\right]\\ -12A_{\delta_{0,g-1}}+(12A_{\delta_{01}}-A_{\delta_{1,1}})=0,\end{aligned}$$

from (S15)

$$(8g^2-26g+20)A_{\kappa_1^2}+(2g-4)A_{\kappa_2}+(4-2g)A_{\delta_1^2}+8(g-1)(g-2)A_{\delta_0^2}=0,$$

from (S16) for $\lfloor g/2 \rfloor \le i \le g-3$

$$\begin{aligned}(4i-1)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(i)}} - A_{\omega^{(i+1)}} + A_{\delta_1^2} + (2i-1)A_{\delta_{1,g-i-1}} \\ = \frac{1}{2i-2} \sum_{\substack{0\le\alpha_0\le\alpha_1\le k-1 \\ \alpha_0+\alpha_1=g-i-1}} m_{i,k,(\alpha_0,\alpha_1)} \cdot N_{g-i-1,k,(k-1-\alpha_1,k-1-\alpha_0)}\end{aligned}$$

and

$$(4g-9)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(g-2)}} + (4g-8)A_{\delta_1^2} + (2g-5)A_{\delta_{1,1}} = \frac{m_{g-2,k,(0,1)}}{2g-6},$$

from (S17)

$$3A_{\kappa_1^2} + A_{\kappa_2} - 2A_{\lambda\delta_0} + A_{\lambda\delta_1} - 44A_{\delta_0^2} - A_{\delta_1^2} + 12A_{\delta_{0,g-1}} - 12A_{\delta_{00}} + A_{\theta_1} = 0,$$

from (S18) for $4 \le i \le \lfloor (g+1)/2 \rfloor$

$$\begin{aligned}3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(i)}} - A_{\omega^{(g-i+1)}} - A_{\delta_1^2} + A_{\delta_{i-1,g-i}} - A_{\lambda^{(i)}} - A_{\lambda^{(g-i+1)}} \\ + A_{\lambda\delta_1} - 12A_{\delta_{0,i-1}} - 12A_{\delta_{0,g-i}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_{i-1}} = 0,\end{aligned}$$

and

$$\begin{aligned}3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(3)}} - A_{\omega^{(g-2)}} - A_{\delta_1^2} + A_{\delta_{2,g-3}} - A_{\lambda^{(3)}} - A_{\lambda\delta_2} \\ + A_{\lambda\delta_1} - 12A_{\delta_{0,2}} - 12A_{\delta_{0,g-3}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_2} = 0,\end{aligned}$$

$$\begin{aligned}3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(2)}} + A_{\delta_{1,g-2}} \\ - A_{\lambda\delta_2} - 12A_{\delta_{0,1}} - 12A_{\delta_{0,g-2}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_1} = 0.\end{aligned}$$

Let $Q_g$ be the square matrix of order $\lfloor (g^2-1)/4 \rfloor + 3g - 1$ associated to the linear system given by the above relations. As we have already noted, since the test surfaces in (S1)-(S18) are defined also for odd values of $g \ge 6$, the matrix $Q_g$ is defined also for $g$ odd, $g \ge 7$. One checks that $|\det(Q_6)| \ne 0$, $|\det(Q_7)| \ne 0$ and for $8 \le g \le 200$, one has

$$\left| \frac{\det(Q_g)}{\det(Q_{g-1})} \right| = \frac{2(g-1)(g-2)(g-4)(13+11(-1)^{g+1})}{(g-3)^2(g-5)}. \tag{2.7.1}$$

It follows that $\det(Q_g) \ne 0$ for all $6 \le g \le 200$. In particular, when $g = 2k$, we are able to solve the system and find the coefficients $A$.

THEOREM 2.7.1. *For $3 \leq k \leq 100$, the class of the locus $\overline{\mathcal{M}}^1_{2k,k} \subset \overline{\mathcal{M}}_{2k}$ is*

$$\begin{aligned}\left[\overline{\mathcal{M}}^1_{2k,k}\right]_Q &= c\Bigg[A_{\kappa_1^2}\kappa_1^2 + A_{\kappa_2}\kappa_2 + A_{\delta_0^2}\delta_0^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\delta_1^2}\delta_1^2 + A_{\lambda\delta_1}\lambda\delta_1 \\ &\quad + A_{\lambda\delta_2}\lambda\delta_2 + \sum_{i=2}^{g-2} A_{\omega^{(i)}}\omega^{(i)} + \sum_{i=3}^{g-3} A_{\lambda^{(i)}}\lambda^{(i)} + \sum_{i,j} A_{\delta_{ij}}\delta_{ij} \\ &\quad + \sum_{i=1}^{\lfloor (g-1)/2 \rfloor} A_{\theta_i}\theta_i\Bigg]\end{aligned}$$

*in $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$, where*

$$\begin{aligned}c &= \frac{2^{k-6}(2k-7)!!}{3(k!)} \\ A_{\kappa_1^2} = -A_{\delta_0^2} &= 3k^2+3k+5 \\ A_{\kappa_2} &= -24k(k+5) \\ A_{\delta_1^2} &= -(3k(9k+41)+5) \\ A_{\lambda\delta_0} &= -24(3(k-1)k-5) \\ A_{\lambda\delta_1} &= 24\left(-33k^2+39k+65\right) \\ A_{\lambda\delta_2} &= 24(3(37-23k)k+185) \\ A_{\omega^{(i)}} &= -180i^4+120i^3(6k+1)-36i^2\left(20k^2+24k-5\right) \\ &\quad +24i\left(52k^2-16k-5\right)+27k^2+123k+5 \\ A_{\lambda^{(i)}} &= 24[6i^2(3k+5)-6i\left(6k^2+23k+5\right) \\ &\quad +159k^2+63k+5] \\ A_{\theta^{(i)}} &= -12i[5i^3+i^2(10-20k)+i\left(20k^2-8k-5\right) \\ &\quad -24k^2+32k-10] \\ A_{\delta_{1,1}} &= 48\left(19k^2-49k+30\right) \\ A_{\delta_{1,g-2}} &= \frac{2}{5}(3k(859k-2453)+2135) \\ A_{\delta_{00}} &= 24k(k-1) \\ A_{\delta_{0,g-2}} &= \frac{2}{5}(3k(187k-389)-745) \\ A_{\delta_{0,g-1}} &= 2(k(31k-49)-65)\end{aligned}$$

*and for $i \geq 1$ and $2 \leq j \leq g-3$*

$$\begin{aligned}A_{\delta_{ij}} &= 2[3k^2(144ij-1)-3k(72ij(i+j+4)+1) \\ &\quad +180i(i+1)j(j+1)-5]\end{aligned}$$

*while*

$$A_{\delta_{0j}} = 2\left(-3\left(12j^2+36j+1\right)k+(72j-3)k^2-5\right)$$

*for* $1 \leq j \leq g-3$.

As usual, for a positive integer $n$, the symbol $(2n+1)!!$ denotes

$$\frac{(2n+1)!}{2^n \cdot n!},$$

while $(-1)!! = 1$.

We conjecture that the formulae in Thm. 2.7.1 represent the class of $\overline{\mathcal{M}}^1_{2k,k}$ also for $k > 100$ (hence for every $k \geq 3$). Since such coefficients $A$ verify all relations in (S1)-(S18), it is enough to show that $\det Q_g \neq 0$ for every $g \geq 6$. To do this, for instance one could show that (2.7.1) holds for every $g \geq 8$.

## 2.8. Pull-back to $\overline{\mathcal{M}}_{2,1}$

As a check, in this section we obtain four more relations for the coefficients $A$ considering the pull-back of $\overline{\mathcal{M}}^1_{2k,k}$ to $\overline{\mathcal{M}}_{2,1}$. Let $j\colon \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ be the map obtained by attaching at elements $(D,p)$ in $\overline{\mathcal{M}}_{2,1}$ a fixed general pointed curve of genus $g-2$. This produces a map $j^*\colon A^2(\overline{\mathcal{M}}_g) \to A^2(\overline{\mathcal{M}}_{2,1})$.

In [**Fab88**, Chapter 3 §1] it is shown that $A^2(\overline{\mathcal{M}}_{2,1})$ has rank 5 and is generated by the classes of the loci composed by curves of type $\Delta_{00}, (a), (b), (c)$ and $(d)$ as in Fig. 2.8.1.

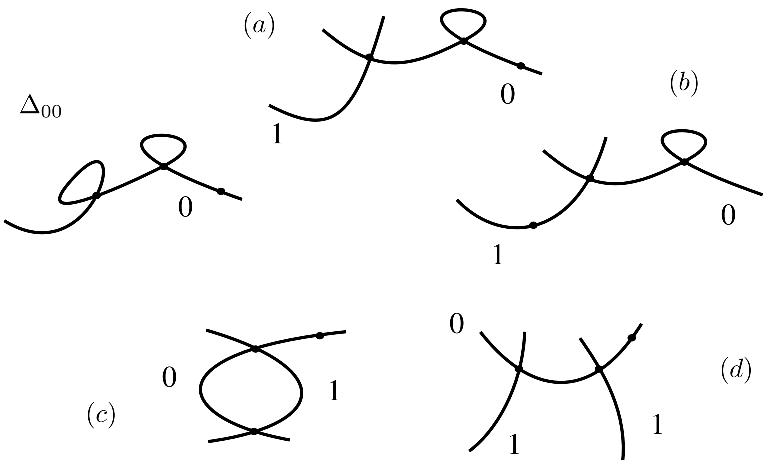



FIGURE 2.8.1. Loci in $\overline{\mathcal{M}}_{2,1}$

We have the following pull-backs

$$
\begin{aligned}
j^*(\delta_{0,1}) &= [(a)]_Q \\
j^*(\delta_{0,g-1}) &= [(b)]_Q \\
j^*(\theta_1) &= [(c)]_Q \\
j^*(\delta_{1,1}) &= [(d)]_Q \\
j^*(\delta_{00}) &= [\Delta_{00}]_Q \\
j^*(\delta_0^2) &= \frac{5}{3}[\Delta_{00}]_Q - 2[(a)]_Q - 2[(b)]_Q \\
j^*(\delta_1^2) &= -\frac{1}{12}\left([(a)]_Q + [(b)]_Q\right) \\
j^*(\lambda\delta_0) &= \frac{1}{6}[\Delta_{00}]_Q \\
j^*(\lambda\delta_1) &= \frac{1}{12}\left([(a)]_Q + [(b)]_Q\right) \\
j^*(\lambda\delta_2) &= -\lambda\psi \\
&= \frac{1}{60}(-[\Delta_{00}]_Q - 7[(a)]_Q - 12[(c)]_Q - 24[(d)]_Q) \\
j^*(\kappa_1^2) &= \left(\frac{1}{5}\delta_0 + \frac{7}{5}\delta_1 + \psi\right)^2 \\
&= \frac{1}{120}(17[\Delta_{00}]_Q + 127[(a)]_Q + 37[(b)]_Q + 120[(c)]_Q \\
&\quad + 840[(d)]_Q) \\
j^*(\kappa_2) &= \lambda(\lambda + \delta_1) + \psi^2 \\
&= \frac{1}{120}(3[\Delta_{00}]_Q + 25[(a)]_Q + 11[(b)]_Q + 24[(c)]_Q + 168[(d)]_Q) \\
j^*(\delta_{1,g-2}) &= -\delta_1\psi \\
&= -\frac{1}{12}[(a)]_Q - 2[(d)]_Q \\
j^*(\delta_{0,g-2}) &= -\delta_0\psi \\
&= -\frac{1}{6}[\Delta_{00}]_Q - [(a)]_Q - 2[(c)]_Q \\
j^*(\omega^{(2)}) &= -\psi^2 \\
&= -\frac{1}{120}([\Delta_{00}]_Q + 13[(a)]_Q - [(b)]_Q + 24[(c)]_Q + 168[(d)]_Q).
\end{aligned}
$$

For this, see relations in [**Fab88**, Chapter 3 §1] and [**Mum83**, §8 - §10]. We have used that on $\overline{\mathcal{M}}_{g,1}$ one has

$$\kappa_i = \kappa_i|_{\overline{\mathcal{M}}_g} + \psi^i$$

(see [**AC96**, 1.10]).

All the other classes have zero pull-back. Finally, $j^*(\overline{\mathcal{M}}^1_{2k,k})$ is supported at most on the locus $(c)$. Indeed general elements in the loci $\Delta_{00}, (a), (b)$ and $(d)$ does not admit any linear series $\mathfrak{g}^1_k$ with adjusted Brill-Noether number less than $-1$ (see also [**Edi93**, Lemma 5.1]). Since the restriction of $\overline{\mathcal{M}}^1_{2k,k}$ to $j(\overline{\mathcal{M}}_{2,1})$ is supported in codimension two, then $j(\overline{\mathcal{M}}_{2,1} \setminus (c)) = 0$. Hence looking at the coefficient of $[\Delta_{00}]_Q$, $[(a)]_Q$, $[(b)]_Q$ and $[(d)]_Q$ in $j^*(\overline{\mathcal{M}}^1_{2k,k})$ we obtain the following relations

$$A_{\delta_{00}} + \frac{5}{3}A_{\delta_0^2} + \frac{1}{6}A_{\lambda\delta_0} - \frac{1}{60}A_{\lambda\delta_2} + \frac{17}{120}A_{\kappa_1^2} + \frac{1}{40}A_{\kappa_2} - \frac{1}{6}A_{\delta_{0,g-2}} - \frac{1}{120}A_{\omega^{(2)}} = 0$$

$$A_{\delta_{01}} - 2A_{\delta_0^2} - \frac{1}{12}A_{\delta_1^2} + \frac{1}{12}A_{\lambda\delta_1} - \frac{7}{60}A_{\lambda\delta_2} + \frac{127}{120}A_{\kappa_1^2} + \frac{5}{24}A_{\kappa_2} - \frac{1}{12}A_{\delta_{1,g-2}} - A_{\delta_{0,g-2}} - \frac{13}{120}A_{\omega^{(2)}} = 0$$

$$A_{\delta_{0,g-1}} - 2A_{\delta_0^2} - \frac{1}{12}A_{\delta_1^2} + \frac{1}{12}A_{\lambda\delta_1} + \frac{37}{120}A_{\kappa_1^2} + \frac{11}{120}A_{\kappa_2} + \frac{1}{120}A_{\omega^{(2)}} = 0$$

$$A_{\delta_{1,1}} - \frac{2}{5}A_{\lambda\delta_2} + 7A_{\kappa_1^2} + \frac{7}{5}A_{\kappa_2} - 2A_{\delta_{1,g-2}} - \frac{7}{5}A_{\omega^{(2)}} = 0.$$

The coefficients $A$ shown in Thm. 2.7.1 satisfy these relations.

## 2.9. Further relations

In this section we will show how to get further relations for the coefficients $A$ that can be used to produce more tests for our result.

**2.9.1. The coefficients of $\kappa_1^2$ and $\kappa_2$.** One can compute the class of $\mathcal{M}^1_{2k,k}$ in the open $\mathcal{M}_{2k}$ by the methods described by Faber in [**Fab99**]. Let $\mathcal{C}^k_{2k}$ be the $k$-fold fiber product of the universal curve over $\mathcal{M}_{2k}$ and let $\pi_i\colon \mathcal{C}^k_{2k} \to \mathcal{C}_{2k}$ be the map forgetting all but the $i$-th point, for $i = 1, \dots, k$. We define the following tautological classes on $\mathcal{C}^k_{2k}$: $K_i$ is the class of $\pi_i^*(\omega)$, where $\omega$ is the relative dualizing sheaf of the map $\mathcal{C}_{2k} \to \mathcal{M}_{2k}$, and $\Delta_{i,j}$ is the class of the locus of curves with $k$ points $(C, x_1, \dots, x_k)$ such that $x_i = x_j$, for $1 \le i, j \le k$.

Let $\mathbb{E}$ be the pull-back to $\mathcal{C}^k_{2k}$ of the Hodge bundle of rank $2k$ and let $\mathbb{F}_k$ be the bundle on $\mathcal{C}^k_{2k}$ of rank $k$ whose fiber over $(C, x_1, \dots, x_k)$ is

$$H^0(C, K_C/K_C(-x_1 \cdots - x_k)).$$

We consider the locus $X$ in $\mathcal{C}^k_{2k}$ where the evaluation map

$$\varphi_k\colon \mathbb{E} \to \mathbb{F}_k$$

has rank at most $k-1$. Equivalently, $X$ parameterizes curves with $k$ points $(C, x_1, \dots, x_k)$ such that $H^0(C, K_C(-x_1 \cdots - x_k)) \ge k+1$ or, in other terms,

$H^0(C, x_1 + \cdots + x_k) \geq 2$. By Porteous formula, the class of $X$ is

$$[X] = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{k+1} \\ 1 & e_1 & e_2 & \cdots & e_k \\ 0 & 1 & e_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e_2 \\ 0 & \cdots & 0 & 1 & e_1 \end{bmatrix}$$

where the $e_i$'s are the Chern classes of $\mathbb{F}_k - \mathbb{E}$. The total Chern class of $\mathbb{F}_k - \mathbb{E}$ is

$$(1+K_1)(1+K_2-\Delta_{1,2})\cdots(1+K_k-\Delta_{1,k}\cdots-\Delta_{k-1,k})(1-\lambda_1+\lambda_2-\lambda_3+\cdots+\lambda_{2k}).$$

Intersecting the class of $X$ with $\Delta_{1,2}$ we obtain a class that pushes forward via $\pi := \pi_1\pi_2\cdots\pi_k$ to the class of $\mathcal{M}^1_{2k,k}$ with multiplicity $(k-2)!(6k-2)$. We refer the reader to [**Fab99**, §4] for formulae for computing the push-forward $\pi_*$.

For instance, when $k = 3$ one constructs a degeneracy locus $X$ on the 3-fold fiber product of the universal curve over $\mathcal{M}_6$. The class of $X$ is

$$[X] = e_1^4 - 3e_1^2 e_2 + e_2^2 + 2e_1 e_3 - e_4$$

where the $e_i$'s are determined by the following total Chern class

$$(1 + K_1)(1 + K_2 - \Delta_{1,2})(1 + K_3 - \Delta_{1,3} - \Delta_{2,3})(1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + \lambda_6).$$

Upon intersecting the class of $X$ with $\Delta_{1,2}$ and using the following identities

$$\begin{gathered} \Delta_{1,3}\Delta_{2,3} = \Delta_{1,2}\Delta_{1,3} \\ \Delta_{1,2}^2 = -K_1\Delta_{1,2} \qquad \Delta_{1,3}^2 = -K_1\Delta_{1,3} \qquad \Delta_{2,3}^2 = -K_2\Delta_{2,3} \\ K_2\Delta_{1,2} = K_1\Delta_{1,2} \qquad K_3\Delta_{1,3} = K_1\Delta_{1,3} \qquad K_3\Delta_{2,3} = K_2\Delta_{2,3}, \end{gathered}$$

one obtains

$$\begin{aligned} [X]\cdot\Delta_{1,2} \quad = \quad & K_3^4\Delta_{1,2} - 3K_3^3\Delta_{1,2}^2 + 7K_3^2\Delta_{1,2}^3 - 15K_3\Delta_{1,2}^4 + 31\Delta_{1,2}^5 \\ & + 72\Delta_{1,2}\Delta_{2,3}^4 + 172\Delta_{1,3}\Delta_{2,3}^4 - K_3^3\Delta_{1,2}\lambda_1 + 3K_3^2\Delta_{1,2}^2\lambda_1 \\ & - 7K_3\Delta_{1,2}^3\lambda_1 + 15\Delta_{1,2}^4\lambda_1 + 23\Delta_{1,2}\Delta_{2,3}^3\lambda_1 + 41\Delta_{1,3}\Delta_{2,3}^3\lambda_1 \\ & + K_3^2\Delta_{1,2}\lambda_1^2 - 3K_3\Delta_{1,2}^2\lambda_1^2 + 7\Delta_{1,2}^3\lambda_1^2 + 6\Delta_{1,2}\Delta_{2,3}^2\lambda_1^2 \\ & + 8\Delta_{1,3}\Delta_{2,3}^2\lambda_1^2 - K_3\Delta_{1,2}\lambda_1^3 + 3\Delta_{1,2}^2\lambda_1^3 + \Delta_{1,2}\Delta_{2,3}\lambda_1^3 \\ & + \Delta_{1,3}\Delta_{2,3}\lambda_1^3 + \Delta_{1,2}\lambda_1^4 - K_3^2\Delta_{1,2}\lambda_2 + 3K_3\Delta_{1,2}^2\lambda_2 - 7\Delta_{1,2}^3\lambda_2 \\ & - 6\Delta_{1,2}\Delta_{2,3}^2\lambda_2 - 8\Delta_{1,3}\Delta_{2,3}^2\lambda_2 + 2K_3\Delta_{1,2}\lambda_1\lambda_2 - 6\Delta_{1,2}^2\lambda_1\lambda_2 \\ & - 2\Delta_{1,2}\Delta_{2,3}\lambda_1\lambda_2 - 2\Delta_{1,3}\Delta_{2,3}\lambda_1\lambda_2 - 3\Delta_{1,2}\lambda_1^2\lambda_2 + \Delta_{1,2}\lambda_2^2 \\ & - K_3\Delta_{1,2}\lambda_3 + 3\Delta_{1,2}^2\lambda_3 + \Delta_{1,2}\Delta_{2,3}\lambda_3 + \Delta_{1,3}\Delta_{2,3}\lambda_3 \\ & + 2\Delta_{1,2}\lambda_1\lambda_3 - \Delta_{1,2}\lambda_4. \end{aligned}$$

Computing the push-forward to $\mathcal{M}_6$ of the above class, one has

$$\begin{aligned}\left[\mathcal{M}^1_{6,3}\right]_Q &= \frac{1}{16}\big((18\kappa_0 - 244)\kappa_2 + 7\kappa_1^2 + (64 - 10\kappa_0)\kappa_1\lambda_1 \\ &\quad + (3\kappa_0^2 - 14\kappa_0)\lambda_1^2 + (14\kappa_0 - 3\kappa_0^2)\lambda_2\big).\end{aligned}$$

Note that $\kappa_0 = 2g - 2 = 10$, $12\lambda_1 = \kappa_1$ and $2\lambda_2 = \lambda_1^2$, hence we recover

$$\left[\mathcal{M}^1_{6,3}\right]_Q = \frac{41}{144}\kappa_1^2 - 4\kappa_2.$$

REMARK 2.9.1. As a corollary one obtains the class of the Maroni locus in $\mathcal{M}_6$. The trigonal locus in $\mathcal{M}_{2k}$ has a divisor known as the Maroni locus (see [**Mar46**], [**MS86**]). While the general trigonal curve of even genus admits an embedding in $\mathbb{P}^1 \times \mathbb{P}^1$ or in $\mathbb{P}^2$ blown up in one point, the trigonal curves admitting an embedding to other kind of ruled surfaces constitute a subvariety of codimension one inside the trigonal locus.

The class of the Maroni locus in the Picard group of the trigonal locus in $\overline{\mathcal{M}}_{2k}$ has been studied in [**SF00**]. For $k = 3$, one has that the class of the Maroni locus is $8\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}^1_{6,3})$. Knowing the class of the trigonal locus in $\mathcal{M}_6$, one has that the class of the Maroni locus in $\mathcal{M}_6$ is

$$8\lambda\left(\frac{41}{144}\kappa_1^2 - 4\kappa_2\right).$$

**2.9.2. More test surfaces.** One could also consider more test surfaces. For instance one can easily adapt the test surfaces of type $(\varepsilon)$ and $(\kappa)$ from [**Fab90b**, §3]. They are all disjoint from the locus $\overline{\mathcal{M}}^1_{2k,k}$ and produce relations compatible with the ones we have shown.

**2.9.3. The relations for $g = 5$.** As an example, let us consider the case $g = 5$. We know that the tautological ring of $\mathcal{M}_5$ is generated by $\lambda$, that is, there is a non-trivial relation among $\kappa_1^2$ and $\kappa_2$ (see [**Fab99**]). The square matrix $Q_5$ from §2.7 expressing the restriction of the generating classes in $\overline{\mathcal{M}}_5$ to the test surfaces (S1)-(S18) (we have to exclude the relation from (S10) which is defined only for $g \geq 6$), has rank 19, hence showing that the class $\kappa_1^2$ (or the class $\kappa_2$) and the 18 boundary classes in codimension two in $\overline{\mathcal{M}}_5$ are independent.

## 2.10. The hyperelliptic locus in $\overline{\mathcal{M}}_4$

The class of the hyperelliptic locus in $\overline{\mathcal{M}}_4$ has been computed in [**FP05**, Prop. 5]. In this section we will recover the formula by the means of the techniques used so far.

The class will be expressed as a linear combination of the 14 generators for $R^2(\overline{\mathcal{M}}_4)$ from [**Fab90b**]: $\kappa_2$, $\lambda^2$, $\lambda\delta_0$, $\lambda\delta_1$, $\lambda\delta_2$, $\delta_0^2$, $\delta_0\delta_1$, $\delta_1^2$, $\delta_1\delta_2$, $\delta_2^2$, $\delta_{00}$, $\gamma_1$, $\delta_{01a}$

and $\delta_{1,1}$. Remember that there exists one unique relation among these classes, namely

$$\begin{aligned}60\kappa_2 - 810\lambda^2 + 156\lambda\delta_0 + 252\lambda\delta_1 - 3\delta_0^2 - 24\delta_0\delta_1 \\ + 24\delta_1^2 - 9\delta_{00} + 7\delta_{01a} - 12\gamma_1 - 84\delta_{1,1} = 0,\end{aligned}$$

hence $R^2(\overline{\mathcal{M}}_4)$ has rank 13. Write $[\overline{\mathcal{M}}^1_{4,2}]_Q$ as

$$\begin{aligned}\left[\overline{\mathcal{M}}^1_{4,2}\right]_Q &= A_{\kappa_2}\kappa_2 + A_{\lambda^2}\lambda^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\lambda\delta_1}\lambda\delta_1 + A_{\lambda\delta_2}\lambda\delta_2 + A_{\delta_0^2}\delta_0^2 \\ &\quad + A_{\delta_0\delta_1}\delta_0\delta_1 + A_{\delta_1^2}\delta_1^2 + A_{\delta_1\delta_2}\delta_1\delta_2 + A_{\delta_2^2}\delta_2^2 + A_{\delta_{00}}\delta_{00} + A_{\gamma_1}\gamma_1 \\ &\quad + A_{\delta_{01a}}\delta_{01a} + A_{\delta_{1,1}}\delta_{1,1}.\end{aligned}$$

Let us construct 13 independent relations among the coefficients $A$.

The surfaces (S1), (S3), (S5), (S6), (S8), (S12)-(S18) from §2.6 give respectively the following 12 independent relations

$$\begin{aligned}8A_{\delta_2^2} &= 36 \\ 4A_{\delta_2^2} - 2A_{\delta_1\delta_2} &= 12 \\ -4A_{\lambda\delta_1} - 48A_{\delta_0\delta_1} + 8A_{\delta_1^2} - 48A_{\delta_{01a}} &= 0 \\ A_{\lambda\delta_2} - A_{\delta_1\delta_2} &= 0 \\ 2A_{\lambda^2} + 24A_{\lambda\delta_0} - 2A_{\lambda\delta_1} + 288A_{\delta_0^2} - 24A_{\delta_0\delta_1} + 2A_{\delta_1^2} + 144A_{\delta_{00}} + A_{\delta_{1,1}} &= 0 \\ -4A_{\lambda\delta_1} + 3A_{\lambda\delta_2} - 48A_{\delta_0\delta_1} + 8A_{\delta_1^2} - 3A_{\delta_1\delta_2} - 12A_{\delta_{01a}} + 3A_{\delta_{1,1}} &= 0 \\ 8A_{\delta_0^2} - 4A_{\delta_0\delta_1} + 2A_{\delta_1^2} - 2A_{\delta_1\delta_2} + 2A_{\delta_2^2} + 4A_{\delta_{0,0}} + A_{\delta_{1,1}} &= 4 \\ -4A_{\lambda\delta_0} - 96A_{\delta_0^2} + 4A_{\delta_0\delta_1} - 48A_{\delta_{00}} - A_{\delta_{1,1}} - 12A_{\delta_{01a}} &= 0 \\ 48A_{\delta_0^2} - 4A_{\delta_1^2} + 4A_{\kappa_2} &= 0 \\ 16A_{\delta_1^2} - 2A_{\delta_2^2} + 2A_{\kappa_2} + 6A_{\delta_{1,1}} &= 30 \\ -2A_{\lambda\delta_0} + A_{\lambda\delta_1} - 44A_{\delta_0^2} + 12A_{\delta_0\delta_1} - A_{\delta_1^2} + A_{\kappa_2} - 12A_{\delta_{00}} + 12A_{\delta_{01a}} + A_{\gamma_1} &= 0 \\ A_{\delta_1\delta_2} - A_{\lambda\delta_2} + A_{\delta_2^2} + A_{\kappa_2} + 12A_{\delta_{01a}} + 12A_{\gamma_1} &= 0.\end{aligned}$$

Next we look at the pull-back to $\overline{\mathcal{M}}_{2,1}$. The pull-back of the classes $\kappa_2$, $\lambda\delta_0$, $\lambda\delta_1$, $\lambda\delta_2$, $\delta_0^2$, $\delta_1^2$, $\delta_{00}$, $\gamma_1 = \theta_1$, $\delta_{01a} = \delta_{0,g-1}$ and $\delta_{1,1}$ have been computed in §2.8.

Moreover

$$\begin{aligned} j^*(\lambda^2) &= \frac{1}{60}([\Delta_{00}]_Q + [(a)]_Q + [(b)]_Q) \\ j^*(\delta_0\delta_1) &= [(a)]_Q + [(b)]_Q \\ j^*(\delta_1\delta_2) &= -\delta_1\psi \\ &= -\frac{1}{12}[(a)]_Q - 2[(d)]_Q \\ j^*(\delta_2^2) &= \psi^2 \\ &= \frac{1}{120}([\Delta_{00}]_Q + 13[(a)]_Q - [(b)]_Q + 24[(c)]_Q + 168[(d)]_Q). \end{aligned}$$

Considering the coefficient of $[\Delta_{00}]_Q$ yields the following relation

$$A_{\delta_{00}} + \frac{5}{3}A_{\delta_0^2} + \frac{1}{6}A_{\lambda\delta_0} - \frac{1}{60}A_{\lambda\delta_2} + \frac{1}{40}A_{\kappa_2} + \frac{1}{60}A_{\lambda^2} + \frac{1}{120}A_{\delta_2^2} = 0.$$

All in all we get 13 independent relations, and the class of $\overline{\mathcal{M}}^1_{4,2}$ follows

$$\begin{aligned} 2\left[\overline{\mathcal{M}}^1_{4,2}\right]_Q = \; & 27\kappa_2 - 339\lambda^2 + 64\lambda\delta_0 + 90\lambda\delta_1 + 6\lambda\delta_2 - \delta_0^2 - 8\delta_0\delta_1 \\ & + 15\delta_1^2 + 6\delta_1\delta_2 + 9\delta_2^2 - 4\delta_{00} - 6\gamma_1 + 3\delta_{01a} - 36\delta_{1,1}. \end{aligned}$$

3

# Double points of plane models in $\overline{\mathcal{M}}_{6,1}$

The birational geometry of an algebraic variety is encoded in its cone of effective divisors. Nowadays a major problem is to determine the effective cone of moduli spaces of curves.

Let $\mathcal{GP}^1_4$ be the Gieseker-Petri divisor in $\mathcal{M}_6$ given by curves with a pencil $l = (\mathscr{L}, V)$ of degree 4 violating the Petri condition, i.e. such that the product map

$$V \otimes H^0(C, K_C \otimes \mathscr{L}^{-1}) \to H^0(C, K_C)$$

is not injective. The class

$$\left[\overline{\mathcal{GP}}^1_4\right] = 94\lambda - 12\delta_0 - 50\delta_1 - 78\delta_2 - 88\delta_3 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_6)$$

is computed in [**EH87**] where classes of Brill-Noether divisors and Gieseker-Petri divisors are determined for arbitrary genera in order to prove that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

Let $\mathfrak{D}^2_d$ be the divisor in $\mathcal{M}_{g,1}$ defined as the locus of smooth pointed curves $[C, p]$ with a net $\mathfrak{g}^2_d$ of Brill-Noether number 0 mapping $p$ to a double point. That is, for values of $g, d$ such that $g = 3(g - d + 2)$,

$$\mathfrak{D}^2_d := \big\{ [C,p] \in \mathcal{M}_{g,1} \mid \\ \exists\, l \in G^2_d(C) \text{ with } l(-p-x) \in G^1_{d-2}(C) \text{ where } x \in C, x \neq p \big\}.$$

Recently Jensen has shown that $\overline{\mathfrak{D}}^2_6$ and the pull-back of $\overline{\mathcal{GP}}^1_4$ to $\overline{\mathcal{M}}_{6,1}$ generate extremal rays of the pseudoeffective cone of $\overline{\mathcal{M}}_{6,1}$ (see [**Jen10**]). It is thus of interest to determine the class of $\overline{\mathfrak{D}}^2_6$.

THEOREM 3.0.1. *The class of the divisor $\overline{\mathfrak{D}}^2_6 \subset \overline{\mathcal{M}}_{6,1}$ is*

$$\left[\overline{\mathfrak{D}}^2_6\right] = 62\lambda + 4\psi - 8\delta_0 - 30\delta_1 - 52\delta_2 - 60\delta_3 - 54\delta_4 - 34\delta_5 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{6,1}).$$

A mix of a Porteous-type argument, the method of test curves and a pull-back to rational pointed curves will lead to the result. Following a method described in [**Kho07**], we realize $\overline{\mathfrak{D}}^2_d$ in $\mathcal{M}^{\mathrm{irr}}_{g,1}$ as the push-forward of a degeneracy locus of a map of vector bundles over $\mathcal{G}^2_d(\mathcal{M}^{\mathrm{irr}}_{g,1})$. This will give us the coefficients of $\lambda$, $\psi$

and $\delta_0$ for the class of $\overline{\mathfrak{D}}^2_d$ in general. Intersecting $\overline{\mathfrak{D}}^2_d$ with carefully chosen one-dimensional families of curves will produce relations to determine the coefficients of $\delta_1$ and $\delta_{g-1}$. Finally in the case $g = 6$ we will get enough relations to find the other coefficients by pulling-back to the moduli space of stable pointed rational curves in the spirit of [**EH87**, §3].

To complete our computation we obtain a general result on some families of linear series on pointed curves with adjusted Brill-Noether number $\rho = 0$ that essentially excludes further ramifications on such families.

THEOREM 3.0.2. *Let $(C, y)$ be a general pointed curve of genus $g > 1$. Let $l$ be a $\mathfrak{g}^r_d$ on $C$ with $r \geq 2$ and adjusted Brill-Noether number $\rho(C, y) = 0$. Denote by $(a_0, a_1, \ldots, a_r)$ the vanishing sequence of $l$ at $y$. Then $l(-a_i y)$ is base-point free for $i = 0, \ldots, r-1$.*

For instance if $C$ is a general curve of genus 4 and $l \in G^2_5(C)$ has vanishing sequence $(0, 1, 3)$ at a general point $p$ in $C$, then $l(-p)$ is base-point free.

Using the irreducibility of the families of linear series with adjusted Brill-Noether number $-1$ ([**EH89**]), we get a similar statement for an arbitrary point on the general curve in such families.

THEOREM 3.0.3. *Let $C$ be a general curve of genus $g > 2$. Let $l$ be a $\mathfrak{g}^r_d$ on $C$ with $r \geq 2$ and adjusted Brill-Noether number $\rho(C, y) = -1$ at an arbitrary point $y$. Denote by $(a_0, a_1, \ldots, a_r)$ the vanishing sequence of $l$ at $y$. Then $l(-a_1 y)$ is base-point free.*

As a verification of Thm. 3.0.1, let us note that the class of $\overline{\mathfrak{D}}^2_6$ is not a linear combination of the class of the Gieseker-Petri divisor $\mathcal{GP}^1_4$ and the class of the divisor $\mathcal{W}$ of Weierstrass points computed in [**Cuk89**]

$$[\mathcal{W}] = -\lambda + 21\psi - 15\delta_1 - 10\delta_2 - 6\delta_3 - 3\delta_4 - \delta_5 \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{6,1}).$$

We prove Thm. 3.0.2 and Thm. 3.0.3 in §3.1. Finally in §3.2 we prove a general version of Thm. 3.0.1. We refer the reader to §1.3 and §2.5 for an introduction to enumerative geometry on the general curve and limit linear series.

## 3.1. Ramifications on some families of linear series with $\rho = 0$ or $-1$

Here we prove Thm. 3.0.2. The result will be repeatedly used in the next section.

PROOF OF THM. 3.0.2. Clearly it is enough to prove the statement for $i = r-1$. We proceed by contradiction. Suppose that for $(C, y)$ a general pointed curve of genus $g$, there exists $x \in C$ such that $h^0(l(-a_{r-1}y - x)) \geq 2$, for some $l$ a $\mathfrak{g}^r_d$ with $\rho(C, y) = 0$. Let us degenerate $C$ to a transversal union $C_1 \cup_{y_1} E_1$, where $C_1$ has genus $g-1$ and $E_1$ is an elliptic curve. Since $y$ is a general point, we can assume $y \in E_1$ and $y - y_1$ not to be a $d!$-torsion point in $\mathrm{Pic}^0(E_1)$.

Let $\{l_{C_1}, l_{E_1}\}$ be a limit $\mathfrak{g}^r_d$ on $C_1 \cup_{y_1} E_1$ such that $a^{l_{E_1}}(y) = (a_0, a_1, \dots, a_r)$. Denote by $(\alpha_0, \dots, \alpha_r)$ the corresponding ramification sequence. We have that $\rho(C_1, y_1) = \rho(E_1, y, y_1) = 0$, hence $w^{l_{C_1}}(y_1) = r + \rho$, where $\rho = \rho(g, r, d)$. Denote by $(b^1_0, b^1_1, \dots, b^1_r)$ the vanishing sequence of $l_{C_1}$ at $y_1$ and by $(\beta^1_0, \beta^1_1, \dots, \beta^1_r)$ the corresponding ramification sequence.

Suppose $x$ specializes to $E_1$. Then $b^1_r \geq a_r + 1$, $b^1_{r-1} \geq a_{r-1} + 1$ and we cannot have both equalities, since $y - y_1$ is not in $\mathrm{Pic}^0(E_1)[d!]$ (see for instance [**Far00**, Prop. 4.1]). Moreover, as usually $b^1_k \geq a_k$ for $0 \leq k \leq r-2$, and again among these inequalities there cannot be more than one equality. We deduce

$$w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + 3 + r - 2 > w^{l_{E_1}}(y) + r = r + \rho$$

hence a contradiction. We have supposed that $h^0(l(-a_{r-1}y - x)) \geq 2$. Then this pencil degenerates to $l_{E_1}(-a_{r-1}y)$ and to a compatible sub-pencil $l'_{C_1}$ of $l_{C_1}(-x)$. We claim that

$$h^0\left(l_{C_1}\left(-b^1_{r-1}y_1 - x\right)\right) \geq 2.$$

Suppose this is not the case. Then we have $a^{l_{C_1}(-x)}(y_1) \leq (b^1_0, \dots, b^1_{r-2}, b^1_r)$, hence $b^1_r \geq a_r$, $b^1_{r-2} \geq a_{r-1}$ and $b^1_k \geq a_k$, for $0 \leq k \leq r-3$. Among these, we cannot have more than one equality, plus $\beta^1_{r-2} \geq \alpha_{r-1} + 1$ and $\beta^1_{r-1} \geq \beta^1_{r-2} > \alpha_{r-1} \geq \alpha_{r-2}$, hence

$$w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + 1 + r - 1 + \beta^1_{r-1} - \alpha_{r-2} > r + \rho$$

a contradiction.

From our assumptions, we have deduced that for $(C_1, y_1)$ a general pointed curve of genus $g-1$, there exist $l_{C_1}$ a $\mathfrak{g}^r_d$ and $x \in C_1$ such that $\rho(C_1, y_1) = 0$ and $h^0(l_{C_1}(-b^1_{r-1}y_1 - x)) \geq 2$, where $b^1_{r-1}$ is as before.

Then we apply the following recursive argument. At the step $i$, we degenerate the pointed curve $(C_i, y_i)$ of genus $g-i$ to a transversal union $C_{i+1} \cup_{y_{i+1}} E_{i+1}$, where $C_{i+1}$ is a curve of genus $g-i-1$ and $E_{i+1}$ is an elliptic curve, such that $y_i \in E_{i+1}$. Let $\{l_{C_{i+1}}, l_{E_{i+1}}\}$ be a limit $\mathfrak{g}^r_d$ on $C_{i+1} \cup_{y_{i+1}} E_{i+1}$ such that $a^{l_{E_{i+1}}}(y_i) = (b^i_0, b^i_1, \dots, b^i_r)$. From $\rho(C_{i+1}, y_{i+1}) = \rho(E_{i+1}, y_i, y_{i+1}) = 0$, we compute that $w^{l_{C_{i+1}}}(y_{i+1}) = (i+1)r + \rho$. Denote by $(b^{i+1}_0, b^{i+1}_1, \dots, b^{i+1}_r)$ the vanishing sequence of $l_{C_{i+1}}$ at $y_{i+1}$. As before we arrive to a contradiction if $x \in E_{i+1}$, and we deduce

$$h^0\left(l_{C_{i+1}}\left(-b^{i+1}_{r-1}y_{i+1} - x\right)\right) \geq 2.$$

At the step $g-2$, our degeneration produces two elliptic curves $C_{g-1} \cup_{y_{g-1}} E_{g-1}$, with $y_{g-2} \in E_{g-1}$. Our assumptions yield the existence of $x \in C_{g-1}$ such that

$$h^0(l_{C_{g-1}}(-b^{g-1}_{r-1}y_{g-1} - x)) \geq 2.$$

We compute $w^{l_{C_{i+1}}}(y_{g-1}) = (g-1)r + \rho$. By the numerical hypothesis, we see that $(g-1)r + \rho = (d-r-1)(r+1) + 1$, hence the vanishing sequence of $l_{C_{g-1}}$ at $y_{g-1}$ has to be $(d-r-1, \dots, d-3, d-2, d)$, whence the contradiction. □

The following proves the similar result for some families of linear series with Brill-Noether number $-1$.

PROOF OF THM 3.0.3. The statement says that for every $y \in C$ such that $\rho(C, y) = -1$ for some $l$ a $\mathfrak{g}^r_d$, and for every $x \in C$, we have that $h^0(l(-a_1 y - x)) \leq r-1$. This is a closed condition and, using the irreducibility of the divisor $\mathcal{D}$ of pointed curves admitting a linear series $\mathfrak{g}^r_d$ with adjusted Brill-Noether number $-1$, it is enough to prove it for $[C, y]$ general in $\mathcal{D}$.

We proceed by contradiction. Suppose for $[C, y]$ general in $\mathcal{D}$ there exists $x \in C$ such that $h^0(l(-a_1 y - x)) \geq r$ for some $l$ a $\mathfrak{g}^r_d$ with $\rho(C, y) = -1$. Let us degenerate $C$ to a transversal union $C_1 \cup_{y_1} E_1$ where $C_1$ is a general curve of genus $g-1$ and $E_1$ is an elliptic curve. Since $y$ is a general point, we can assume $y \in E_1$. Let $\{l_{C_1}, l_{E_1}\}$ be a limit $\mathfrak{g}^r_d$ on $C_1 \cup_{y_1} E_1$ such that $a^{l_{E_1}}(y) = (a_0, a_1, \ldots, a_r)$. Then $\rho(E_1, y, y_1) \leq -1$ and $\rho(C_1, y_1) = 0$, hence $w^{l_{C_1}}(y_1) = r + \rho$ (see also [**Far09b**, Proof of Thm. 4.6]). Let $(b^1_0, b^1_1, \ldots, b^1_r)$ be the vanishing sequence of $l_{C_1}$ at $y_1$ and $(\beta^1_0, \beta^1_1, \ldots, \beta^1_r)$ the corresponding ramification sequence.

The point $x$ has to specialize to $C_1$. Indeed suppose $x \in E_1$. Then $b^1_k \geq a_k + 1$ for $k \geq 1$. This implies $w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + r > \rho + r$, hence a contradiction. Then $x \in C_1$, and $l(-a_1 y - x)$ degenerates to $l_{E_1}(-a_1 y)$ and to a compatible system $l'_{C_1} := l_{C_1}(-x)$. We claim that

$$h^0\left(l_{C_1}\left(-b^1_{r-1} y_1 - x\right)\right) \geq 2.$$

Suppose this is not the case. Then we have $a^{l'_{C_1}}(y_1) \leq (b^1_0, \ldots, b^1_{r-2}, b^1_r)$ and so $b^1_r \geq a_r$, and $b^1_k \geq a_{k+1}$ for $0 \leq k \leq r-2$. Then $\beta^1_k \geq \alpha_{k+1} + 1$ for $k \leq r-2$, and summing up we obtain

$$w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y) + r - 1 + \beta^1_{r-1} - \alpha_0.$$

Clearly $\beta^1_{r-1} \geq \beta^1_{r-2} > \alpha_{r-1} \geq \alpha_0$. Hence $w^{l_{C_1}}(y_1) > \rho + r$, a contradiction.

All in all from our assumptions we have deduced that for a general pointed curve $(C_1, y_1)$ of genus $g-1$, there exist $l_{C_1}$ a $\mathfrak{g}^r_d$ and $x \in C_1$ such that $\rho(C_1, y_1) = 0$ and $h^0(l_{C_1}(-b^1_{r-1} y_1 - x)) \geq 2$, where $b^1_{r-1}$ is as before. This contradicts Thm. 3.0.2, hence we obtain the statement. □

## 3.2. The divisor $\mathfrak{D}^2_d$

Remember that $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$ is generated by the Hodge class $\lambda$, the cotangent class $\psi$ corresponding to the marked point, and the boundary classes $\delta_0, \ldots \delta_{g-1}$ defined as follows. The class $\delta_0$ is the class of the closure of the locus of pointed irreducible nodal curves, and the class $\delta_i$ is the class of the closure of the locus of pointed curves $[C_i \cup C_{g-i}, p]$ where $C_i$ and $C_{g-i}$ are smooth curves respectively of genus $i$ and $g-i$ meeting transversally in one point, and $p$ is a smooth point in $C_i$, for $i = 1, \ldots, g-1$. In this section we prove the following theorem.

THEOREM 3.2.1. *Let $g = 3s$ and $d = 2s + 2$ for $s \geq 1$. The class of the divisor $\overline{\mathfrak{D}}_d^2$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$ is*

$$\left[\overline{\mathfrak{D}}_d^2\right] = a\lambda + c\psi - \sum_{i=0}^{g-1} b_i\delta_i$$

*where*

$$\begin{aligned} a &= \frac{48s^4 + 80s^3 - 16s^2 - 64s + 24}{(3s-1)(3s-2)(s+3)} N_{g,2,d} \\ c &= \frac{2s(s-1)}{3s-1} N_{g,2,d} \\ b_0 &= \frac{24s^4 + 23s^3 - 18s^2 - 11s + 6}{3(3s-1)(3s-2)(s+3)} N_{g,2,d} \\ b_1 &= \frac{14s^3 + 6s^2 - 8s}{(3s-2)(s+3)} N_{g,2,d} \\ b_{g-1} &= \frac{48s^4 + 12s^3 - 56s^2 + 20s}{(3s-1)(3s-2)(s+3)} N_{g,2,d}. \end{aligned}$$

*Moreover for $g = 6$ and for $i = 2, 3, 4$, we have that*

$$b_i = -7i^2 + 43i - 6.$$

**3.2.1. The coefficient $c$.** The coefficient $c$ can be quickly found. Let $C$ be a general curve of genus $g$ and consider the curve $\overline{C} = \{[C, y] : y \in C\}$ in $\overline{\mathcal{M}}_{g,1}$ obtained varying the point $y$ on $C$. Then the only generator class having non-zero intersection with $\overline{C}$ is $\psi$, and $\overline{C} \cdot \psi = 2g - 2$. On the other hand, $\overline{C} \cdot \overline{\mathfrak{D}}_d^2$ is equal to the number of triples $(x, y, l) \in C \times C \times G_d^2(C)$ such that $x$ and $y$ are different points and $h^0(l(-x-y)) \geq 2$. The number of such linear series on a general $C$ is computed by the Castelnuovo number (remember that $\rho = 0$), and for each of them the number of couples $(x, y)$ imposing only one condition is twice the number of double points, computed by the Plücker formula. Hence we get the equation

$$\overline{\mathfrak{D}}_d^2 \cdot \overline{C} = 2\left(\frac{(d-1)(d-2)}{2} - g\right) N_{g,2,d} = c\,(2g-2)$$

and so

$$c = \frac{2s(s-1)}{3s-1} N_{g,2,d}.$$

**3.2.2. The coefficients $a$ and $b_0$.** In order to compute $a$ and $b_0$, we use a Porteous-style argument. Let $\mathcal{G}_d^2$ be the family parametrizing triples $(C, p, l)$, where $[C, p] \in \mathcal{M}_{g,1}^{\mathrm{irr}}$ and $l$ is a $\mathfrak{g}_d^2$ on $C$; denote by $\eta\colon \mathcal{G}_d^2 \to \mathcal{M}_{g,1}^{\mathrm{irr}}$ the natural map. There exists $\pi\colon \mathcal{Y}_d^2 \to \mathcal{G}_d^2$ a universal pointed quasi-stable curve, with $\sigma\colon \mathcal{G}_d^2 \to \mathcal{Y}_d^2$

the marked section. Let $\mathscr{L} \to \mathcal{Y}^2_d$ be the universal line bundle of relative degree $d$ together with the trivialization $\sigma^*(\mathscr{L}) \cong \mathscr{O}_{\mathcal{G}^2_d}$, and $\mathscr{V} \subset \pi_*(\mathscr{L})$ be the sub-bundle which over each point $(C, p, l = (L, V))$ in $\mathcal{G}^2_d$ restricts to $V$. (See [**Kho07**, §2] for more details.)

Furthermore let us denote by $\mathcal{Z}^2_d$ the family parametrizing

$$((C, p), x_1, x_2, l)\,,$$

where $[C, p] \in \mathcal{M}^{\text{irr}}_{g,1}$, $x_1, x_2 \in C$ and $l$ is a $\mathfrak{g}^2_d$ on $C$, and let $\mu, \nu\colon \mathcal{Z}^2_d \to \mathcal{Y}^2_d$ be defined as the maps that send $((C, p), x_1, x_2, l)$ respectively to $((C, p), x_1, l)$ and $((C, p), x_2, l)$.

Now given a linear series $l = (L, V)$, the natural map

$$\varphi\colon V \to H^0(L|_{p+x})$$

globalizes to

$$\widetilde{\varphi}\colon \mathscr{V} \to \mu_*\left(\nu^*\mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\Gamma_\sigma+\Delta}\right) =: \mathscr{M}$$

as a map of vector bundles over $\mathcal{Y}^2_d$, where $\Delta$ and $\Gamma_\sigma$ are the loci in $\mathcal{Z}^2_d$ determined respectively by $x_1 = x_2$ and $x_2 = p$. Then $\overline{\mathfrak{D}}^2_d \cap \mathcal{M}^{\text{irr}}_{g,1}$ is the push-forward of the locus in $\mathcal{Y}^2_d$ where $\widetilde{\varphi}$ has rank $\leq 1$. Using Porteous formula, we have

$$\begin{aligned}
(3.2.1) \qquad \left[\overline{\mathfrak{D}}^2_d\right]\Big|_{\mathcal{M}^{\text{irr}}_{g,1}} &= \eta_*\pi_*\left[\frac{\mathscr{V}^\vee}{\mathscr{M}^\vee}\right]_2 \\
&= \eta_*\pi_*\big(\pi^*c_2(\mathscr{V}^\vee) + \pi^*c_1(\mathscr{V}^\vee)\cdot c_1(\mathscr{M}) \\
&\quad + c_1^2(\mathscr{M}) - c_2(\mathscr{M})\big).
\end{aligned}$$

Let us find the Chern classes of $\mathscr{M}$. Tensoring the exact sequence

$$0 \to \mathscr{I}_\Delta/\mathscr{I}_{\Delta+\Gamma_\sigma} \to \mathscr{O}/\mathscr{I}_{\Delta+\Gamma_\sigma} \to \mathscr{O}_\Delta \to 0$$

by $\nu^*\mathscr{L}$ and applying $\mu_*$, we deduce that

$$\begin{aligned}
ch(\mathscr{M}) &= ch(\mu_*(\mathscr{O}_{\Gamma_\sigma}(-\Delta) \otimes \nu^*\mathscr{L})) + ch(\mu_*(\mathscr{O}_\Delta \otimes \nu^*\mathscr{L})) \\
&= ch(\mu_*(\mathscr{O}_{\Gamma_\sigma}(-\Delta))) + ch(\mu_*(\mathscr{O}_\Delta \otimes \nu^*\mathscr{L})) \\
&= e^{-\sigma} + ch(\mathscr{L})
\end{aligned}$$

hence

$$\begin{aligned}
c_1(\mathscr{M}) &= c_1(\mathscr{L}) - \sigma \\
c_2(\mathscr{M}) &= -\sigma c_1(\mathscr{L}).
\end{aligned}$$

The following classes

$$\begin{aligned}
\alpha &= \pi_*\left(c_1(\mathscr{L})^2 \cap \left[\mathcal{Y}^2_d\right]\right) \\
\gamma &= c_1(\mathscr{V}) \cap \left[\mathcal{G}^2_d\right]
\end{aligned}$$

have been studied in [**Kho07**, Thm. 2.11]. In particular

$$\begin{aligned}\frac{6(g-1)(g-2)}{dN_{g,2,d}}\eta_*(\alpha)|_{\mathcal{M}_{g,1}^{\mathrm{irr}}} &= 6(gd-2g^2+8d-8g+4)\lambda \\ &\quad +(2g^2-gd+3g-4d-2)\delta_0 \\ &\quad -6d(g-2)\psi, \\ \frac{2(g-1)(g-2)}{N_{g,2,d}}\eta_*(\gamma)|_{\mathcal{M}_{g,1}^{\mathrm{irr}}} &= (-(g+3)\xi+40)\,\lambda \\ &\quad +\frac{1}{6}\left((g+1)\xi-24\right)\delta_0 \\ &\quad -3d(g-2)\psi,\end{aligned}$$

where

$$\xi = 3(g-1)+\frac{(g+3)(3g-2d-1)}{g-d+5}.$$

Plugging into (3.2.1) and using the projection formula, we find

$$\begin{aligned}\left[\overline{\mathfrak{D}}_d^2\right]\Big|_{\mathcal{M}_{g,1}^{\mathrm{irr}}} &= \eta_*\left(-\gamma\cdot\pi_*c_1(\mathscr{L})+\gamma\cdot\pi_*\sigma+\alpha+\pi_*\sigma^2-\pi_*(\sigma c_1(\mathscr{L}))\right) \\ &= (1-d)\eta_*(\gamma)+\eta_*(\alpha)-N_{g,2,d}\cdot\psi.\end{aligned}$$

Hence

$$\begin{aligned} a &= \frac{48s^4+80s^3-16s^2-64s+24}{(3s-1)(3s-2)(s+3)}N_{g,2,d} \\ b_0 &= \frac{24s^4+23s^3-18s^2-11s+6}{3(3s-1)(3s-2)(s+3)}N_{g,2,d}\end{aligned}$$

and we recover the previously computed coefficient $c$.

**3.2.3. The coefficient** $b_1$**.** Let $C$ be a general curve of genus $g-1$ and $(E,p,q)$ a two-pointed elliptic curve, with $p-q$ not a torsion point in $\mathrm{Pic}^0(E)$. Let $\overline{C}_1 := \{(C\cup_{y\sim q}E,p)\}_{y\in C}$ be the family of curves obtained identifying the point $q\in E$ with a moving point $y\in C$. Computing the intersection of the divisor $\overline{\mathfrak{D}}_d^2$ with $\overline{C}_1$ is equivalent to answering the following question: how many triples $(x,y,l)$ are there, with $y\in C$, $x\in C\cup_{y\sim q}E\setminus\{p\}$ and $l=\{l_C,l_E\}$ a limit $\mathfrak{g}_d^2$ on $C\cup_{y\sim q}E$, such that $(p,x,l)$ arises as limit of $(p_t,x_t,l_t)$ on a family of curves $\{C_t\}_t$ with smooth general element, where $p_t$ and $x_t$ impose only one condition on $l_t$ a $\mathfrak{g}_d^2$?

Let $a^{l_E}(q)=(a_0,a_1,a_2)$ be the vanishing sequence of $l_E\in G_d^2(E)$ at $q$. Since $C$ is general, there are no $\mathfrak{g}_{d-1}^2$ on $C$, hence $l_C$ is base-point free and $a_2=d$. Moreover we know $a_1\leq d-2$. Let us suppose $x\in E\setminus\{q\}$. We distinguish two cases. If $\rho(E,q)=\rho(C,y)=0$, then $w^{l_E}(q)=\rho(1,2,d)=3d-8$. Thus $a^{l_E}(q)=(d-3,d-2,d)$. Removing the base point we have that $l_E(-(d-3)q)$ is a $\mathfrak{g}^2_3$ and $l_E(-(d-3)q-p-x)$ produces a $\mathfrak{g}^1_1$ on $E$, hence a contradiction. The other case is $\rho(E,q)=1$ and $\rho(C,y)\leq -1$. These force $a^{l_E}(q)=(d-4,d-2,d)$ and $a^{l_C}(y)\geq (0,2,4)$. On $E$ we have that $l_E(-(d-4)q-p-x)$ is a $\mathfrak{g}^1_2$.

The question splits in two: firstly, how many linear series $l_E\in G^2_4(E)$ and points $x\in E\setminus\{q\}$ are there such that $a^{l_E}(q)=(0,2,4)$ and $l_E(-p-x)\in G^1_2(E)$? The first condition restricts our attention to the linear series $l_E=(\mathscr{O}(4q),V)$ where $V$ is a tridimensional vector space and $H^0(\mathscr{O}(4q-2q))\subset V$, while the second condition tells us $H^0(\mathscr{O}(4q-p-x))\subset V$. If $x=p$, then we get $p-q$ is a torsion point in $\mathrm{Pic}^0(E)$, a contradiction. On the other hand, if $x\in E\setminus\{p,q\}$, then $H^0(\mathscr{O}(4q-2q))\cap H^0(\mathscr{O}(4q-p-x))\neq\emptyset$ entails $p+x\equiv 2q$. Hence the point $x$ and the space $V=H^0(\mathscr{O}(4q-2q))+H^0(\mathscr{O}(4q-p-x))$ are uniquely determined.

Secondly, how many couples $(y,l_C)\in C\times G^2_d(C)$ are there, such that the vanishing sequence of $l_C$ at $y$ is greater than or equal to $(0,2,4)$? This is a particular case of a problem discussed in [**Far09b**, Proof of Thm. 4.6]. The answer is

$$(g-1)\left(15N_{g-1,2,d,(0,2,2)}+3N_{g-1,2,d,(1,1,2)}+3N_{g-1,2,d,(0,1,3)}\right)\\ =\frac{24(2s^2+3s-4)}{s+3}N_{g,2,d}.$$

Now let us suppose $x\in C\setminus\{y\}$. The condition on $x$ and $p$ can be reformulated in the following manner. We consider the curve $C\cup_y E$ as the special fiber $X_0$ of a family of curves $\pi\colon X\to B$ with sections $x(t)$ and $p(t)$ such that $x(0)=x$, $p(0)=p$, and with smooth general fiber having $l=(\mathscr{L},V)$ a $\mathfrak{g}^2_d$ such that $l(-x-p)$ is a $\mathfrak{g}^1_{d-2}$. Let $V'\subset V$ be the two dimensional linear subspace formed by those sections $\sigma\in V$ such that $\mathrm{div}(\sigma)\geq x+p$. Then $V'$ specializes on $X_0$ to $V'_C\subset V_C$ and $V'_E\subset V_E$ two-dimensional subspaces, where $\{l_C=(\mathscr{L}_C,V_C),l_E=(\mathscr{L}_E,V_E)\}$ is a limit $\mathfrak{g}^2_d$, such that

$$\begin{cases}\mathrm{ord}_y(\sigma_C)+\mathrm{ord}_y(\sigma_E)\geq d\\ \mathrm{div}(\sigma_C)\geq x\\ \mathrm{div}(\sigma_E)\geq p\end{cases}$$

for every $\sigma_C\in V'_C$ and $\sigma_E\in V'_E$. Let $l'_C:=(\mathscr{L}_C,V'_C)$ and $l'_E:=(\mathscr{L}_E,V'_E)$. Note that since $\sigma_E\geq p$, we get $\mathrm{ord}_y(\sigma_E)<d$, $\forall\,\sigma_E\in V'_E$. Then $\mathrm{ord}_y(\sigma_C)>0$, hence $\mathrm{ord}_y(\sigma_C)\geq 2$, since $y$ is a cuspidal point on $C$. Removing the base point, $l'_C$ is a $\mathfrak{g}^1_{d-2}$ such that $l'_C(-x)$ is a $\mathfrak{g}^1_{d-3}$. Let us suppose $\rho(E,y)=1$ and $\rho(C,y)=-1$. Then $a^{l_E}(y)=(d-4,d-2,d)$, $a^{l'_E}(y)=(d-4,d-2)$, $a^{l_C}(y)=(0,2,4)$ and $a^{l'_C}(y)=(2,4)$. Now $l_C$ is characterized by the conditions $H^0(l_C(-2y-x))\geq 2$ and $H^0(l_C(-4y-x))\geq 1$. By Thm. 3.0.3 this possibility does not occur.

Suppose now $\rho(E,y)=\rho(C,y)=0$. Then $a^{l_E}(y)=(d-3,d-2,d)$, i.e. $l_E(-(d-3)y)=|3y|$ is uniquely determined. On the $C$ aspect we have that

$a^{l_C}(y) = (0,2,3)$ and $h^0(l_C(-2y-x)) \geq 2$. Hence we are interested on $Y$, the locus of triples $(x, y, l_C)$ such that the map

$$\varphi\colon H^0(l_C) \to H^0(l_C|_{2y+x})$$

has rank $\leq 1$. By Thm. 3.0.2 there is only a finite number of such triples, and clearly the case $a^{l_C}(y) > (0,2,3)$ cannot occur. Moreover, note that $x$ and $y$ will be necessarily distinct.

Let $\mu = \pi_{1,2,4}\colon C\times C\times C\times W^2_d(C) \to C\times C\times W^2_d(C)$ and $\nu = \pi_{3,4}\colon C\times C\times C\times W^2_d(C) \to C\times W^2_d(C)$ be the natural projections respectively on the first, second and fourth components, and on the third and fourth components. Let $\pi\colon C\times C\times W^2_d(C) \to W^2_d(C)$ be the natural projection on the third component. Now $\varphi$ globalizes to

$$\widetilde{\varphi}\colon \pi^*\mathscr{E} \to \mu_*\left(\nu^*\mathscr{L}\otimes\mathscr{O}/\mathscr{I}_{\mathcal{D}}\right) =: \mathscr{M}$$

as a map of rank 3 bundles over $C\times C\times W^2_d(C)$, where $\mathcal{D}$ is the pullback to $C\times C\times C\times W^2_d(C)$ of the divisor on $C\times C\times C$ that on $(x,y,C)\cong C$ restricts to $x+2y$, $\mathscr{L}$ is a Poincaré bundle on $C\times W^2_d$ and $\mathscr{E}$ is the push-forward of $\mathscr{L}$ to $W^2_d(C)$. Then $Y$ is the degeneracy locus where $\widetilde{\varphi}$ has rank $\leq 1$. Let $\mathfrak{c}_i := c_i(\mathscr{E})$ be the Chern classes of $\mathscr{E}$. By Porteous formula, we have

$$[Y] = \begin{bmatrix} e_2 & e_3 \\ e_1 & e_2 \end{bmatrix}$$

where the $e_i$'s are the Chern classes of $\pi^*\mathscr{E}^\vee - \mathscr{M}^\vee$, i.e.

$$\begin{aligned}
e_1 &= \mathfrak{c}_1 + c_1(\mathscr{M})\\
e_2 &= \mathfrak{c}_2 + \mathfrak{c}_1 c_1(\mathscr{M}) + c_1^2(\mathscr{M}) - c_2(\mathscr{M})\\
e_3 &= \mathfrak{c}_3 + \mathfrak{c}_2 c_1(\mathscr{M}) + \mathfrak{c}_1\left(c_1^2(\mathscr{M}) - c_2(\mathscr{M})\right)\\
&\quad + \left(c_1^3(\mathscr{M}) + c_3(\mathscr{M}) - 2c_1(\mathscr{M})c_2(\mathscr{M})\right).
\end{aligned}$$

Let us find the Chern classes of $\mathscr{M}$. First we develop some notation (see also [**ACGH85**, §VIII.2]). Let $\pi_i\colon C\times C\times C\times W^2_d(C) \to C$ for $i=1,2,3$ and $\pi_4\colon C\times C\times C\times W^2_d(C) \to W^2_d(C)$ be the natural projections. Denote by $\theta$ the pull-back to $C\times C\times C\times W^2_d(C)$ of the class $\theta \in H^2(W^2_d(C))$ via $\pi_4$, and denote by $\eta_i$ the cohomology class $\pi_i^*([\text{point}]) \in H^2(C\times C\times C\times W^2_d(C))$, for $i=1,2,3$. Note that $\eta_i^2 = 0$. Furthermore, given a symplectic basis $\delta_1,\dots,\delta_{2(g-1)}$ for $H^1(C,\mathbb{Z}) \cong H^1(W^2_d(C),\mathbb{Z})$, denote by $\delta^i_\alpha$ the pull-back to $C\times C\times C\times W^2_d(C)$ of $\delta_\alpha$ via $\pi_i$, for $i=1,2,3,4$. Let us define

$$\gamma_{ij} := -\sum_{\alpha=1}^{g-1}\left(\delta^j_\alpha\delta^i_{g-1+\alpha} - \delta^j_{g-1+\alpha}\delta^i_\alpha\right).$$

Note that

$$\begin{array}{rclcrclll} \gamma_{ij}^2 & = & -2(g-1)\eta_i\eta_j & \text{and} & \eta_i\gamma_{ij} & = & \gamma_{ij}^3 = 0 & \text{for} & 1 \le i < j \le 3, \\ \gamma_{k4}^2 & = & -2\eta_k\theta & \text{and} & \eta_k\gamma_{k4} & = & \gamma_{k4}^3 = 0 & \text{for} & k = 1,2,3. \end{array}$$

Moreover

$$\gamma_{ij}\gamma_{jk} = \eta_j\gamma_{ik},$$

for $1 \le i < j < k \le 4$. With this notation, we have

$$ch(\nu^*\mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\mathcal{D}}) = (1 + d\eta_3 + \gamma_{34} - \eta_3\theta)\left(1 - e^{-(\eta_1+\gamma_{13}+\eta_3+2\eta_2+2\gamma_{23}+2\eta_3)}\right),$$

hence by Grothendieck-Riemann-Roch

$$\begin{aligned} ch(\mathscr{M}) &= \mu_*\left((1+(2-g)\eta_3)ch(\nu^*\mathscr{L} \otimes \mathscr{O}/\mathscr{I}_{\mathcal{D}})\right) \\ &= 3 + (d-2)\eta_1 + (2g+2d-6)\eta_2 - 2\gamma_{12} + \gamma_{14} + 2\gamma_{24} \\ &\quad - \eta_1\theta - 2\eta_2\theta + (8-2d-4g)\eta_1\eta_2 - 2\eta_1\gamma_{24} - 2\eta_2\gamma_{14} + 2\eta_1\eta_2\theta. \end{aligned}$$

Using Newton's identities, we recover the Chern classes of $\mathscr{M}$:

$$\begin{aligned} c_1(\mathscr{M}) &= (d-2)\eta_1 + (2g+2d-6)\eta_2 - 2\gamma_{12} + \gamma_{14} + 2\gamma_{24}, \\ c_2(\mathscr{M}) &= (2d^2 - 8d + 2gd + 8 - 4g)\eta_1\eta_2 + (2g+2d-8)\eta_2\gamma_{14} \\ &\quad + (2d-4)\eta_1\gamma_{24} + 2\gamma_{14}\gamma_{24} - 2\eta_2\theta, \\ c_3(\mathscr{M}) &= (4-2d)\eta_1\eta_2\theta - 2\eta_2\gamma_{14}\theta. \end{aligned}$$

We finally find

$$\begin{aligned} [Y] &= \eta_1\eta_2(\mathfrak{c}_1^2(2d^2 - 8d + 2dg + 4 - 4(g-1)) \\ &\quad + \mathfrak{c}_1\theta(-12d - 4g + 40) + \mathfrak{c}_2(-4d + 16 - 8g) + 12\theta^2) \\ &= \frac{(28s+48)(s-2)(s-1)}{(s+3)} N_{g,2,d} \cdot \eta_1\eta_2\theta^{g-1} \end{aligned}$$

where we have used the following identities proved in [**Far09b**, Lemma 2.6]

$$\begin{aligned} \mathfrak{c}_1^2 &= \left(1 + \frac{2s+2}{s+3}\right)\mathfrak{c}_2 \\ \mathfrak{c}_1\theta &= (s+1)\mathfrak{c}_2 \\ \theta^2 &= \frac{(s+1)(s+2)}{3}\mathfrak{c}_2 \\ \mathfrak{c}_2 &= N_{g,2,d} \cdot \theta^{g-1}. \end{aligned}$$

We are going to show that we have already considered all non zero contributions. Indeed let us suppose $x = y$. Blowing up the point $x$, we obtain $C \cup_y \mathbb{P}^1 \cup_q E$ with $x \in \mathbb{P}^1 \setminus \{y, q\}$ and $p \in E \setminus \{q\}$. We reformulate the condition on $x$ and $p$ viewing our curve as the special fiber of a family of curves $\pi\colon X \to B$ as before. Let $\{l_C, l_{\mathbb{P}^1}, l_E\}$ be a limit $\mathfrak{g}^2_d$. Now $V'$ specializes to $V'_C$, $V'_{\mathbb{P}^1}$ and $V'_E$. There are three possibilities: either $\rho(C, y) = \rho(\mathbb{P}^1, x, y, q) = \rho(E, p, q) = 0$, or

$\rho(C,y)=-1$, $\rho(\mathbb{P}^1,x,y,q)=0$, $\rho(E,p,q)=1$, or $\rho(C,y)=-1$, $\rho(\mathbb{P}^1,x,y,q)=1$, $\rho(E,p,q)=0$. In all these cases $a^{l_C}(y)=(0,2,a_2^{l_C}(y))$ (remember that $l_C$ is base-point free) and $a^{l_E}(q)=(a_0^{l_E}(q),d-2,d)$. Hence $a^{l_{\mathbb{P}^1}}(y)=(a_0^{l_{\mathbb{P}^1}}(y),d-2,d)$ and $a^{l_{\mathbb{P}^1}}(q)=(0,2,a_2^{l_{\mathbb{P}^1}}(q))$. Let us restrict now to the sections in $V'_C$, $V'_{\mathbb{P}^1}$ and $V'_E$. For all sections $\sigma_{\mathbb{P}^1}\in V'_{\mathbb{P}^1}$ since $\mathrm{div}(\sigma_{\mathbb{P}^1})\geq x$, we have that $\mathrm{ord}_y(\sigma_{\mathbb{P}^1})<d$ and hence $\mathrm{ord}_y(\sigma_{\mathbb{P}^1})\leq d-2$. On the other side, since for all $\sigma_E\in V'_E$, $\mathrm{div}(\sigma_E)\geq p$, we have that $\mathrm{ord}_q(\sigma_E)<d$ and hence $\mathrm{ord}_q(\sigma_{\mathbb{P}^1})\geq 2$. Let us take one section $\tau\in V'_{\mathbb{P}^1}$ such that $\mathrm{ord}_y(\tau)=d-2$. Since $\mathrm{div}(\tau)\geq(d-2)y+x$, we get $\mathrm{ord}_q(\tau)\leq 1$, hence a contradiction.

Thus we have that

$$\overline{\mathfrak{D}}^2_d\cdot\overline{C}_1=\frac{24(2s^2+3s-4)}{s+3}N_{g,2,d}+\frac{(28s+48)(s-2)(s-1)}{(s+3)}N_{g,2,d}.$$

while considering the intersection of the test curve $\overline{C}_1$ with the generating classes we have

$$\overline{\mathfrak{D}}^2_d\cdot\overline{C}_1=b_1(2g-4),$$

whence

$$b_1=\frac{14s^3+6s^2-8s}{(3s-2)(s+3)}N_{g,2,d}.$$

REMARK 3.2.2. The previous class $[Y]$ being nonzero, it implies together with Thm. 3.0.2 that the scheme $\mathcal{G}^2_d((0,2,3))$ over $\mathcal{M}_{g-1,1}$ is not irreducible.

**3.2.4. The coefficient $b_{g-1}$.** We analyze now the following test curve $\overline{E}$. Let $(C,p)$ be a general pointed curve of genus $g-1$ and $(E,q)$ be a pointed elliptic curve. Let us identify the points $p$ and $q$ and let $y$ be a movable point in $E$. We have

$$0=\overline{\mathfrak{D}}^2_d\cdot\overline{E}=c+b_1-b_{g-1},$$

whence

$$b_{g-1}=\frac{48s^4+12s^3-56s^2+20s}{(3s-1)(3s-2)(s+3)}N_{g,2,d}.$$

**3.2.5. A test.** Furthermore, as a test we consider the family of curves $R$. Let $(C,p,q)$ be a general two-pointed curve of genus $g-1$ and let us identify the point $q$ with the base point of a general pencil of plane cubic curves. We have

$$0=\overline{\mathfrak{D}}^2_d\cdot R=a-12b_0+b_{g-1}.$$

**3.2.6. The remaining coefficients in case $g=6$.** Denote by $P_g$ the moduli space of stable $g$-pointed rational curves. Let $(E,p,q)$ be a general two-pointed elliptic curve and let $j\colon P_g\to\overline{\mathcal{M}}_{g,1}$ be the map obtained identifying the first marked point on a rational curve with the point $q\in E$ and attaching a fixed elliptic tail at the other marked points. We claim that $j^*(\overline{\mathfrak{D}}^2_6)=0$.

Indeed consider a flag curve of genus 6 in the image of $j$. Clearly the only possibility for the adjusted Brill-Noether numbers is to be zero on each aspect. In particular the collection of the aspects on all components but $E$ smooths to a $\mathfrak{g}^2_6$ on a general one-pointed curve of genus 5. As discussed in section 3.2.3, the point $x$ can not be in $E$. Suppose $x$ is in the rest of the curve. Then smoothing we get $l$ a $\mathfrak{g}^2_6$ on a general pointed curve of genus 5 such that $l(-2q-x))$ is a $\mathfrak{g}^1_3$, a contradiction.

Now let us study the pull-back of the generating classes. As in [**EH87**, §3] we have that $j^*(\lambda) = j^*(\delta_0) = 0$. Furthermore $j^*(\psi) = 0$.

For $i = 1, \ldots, g-3$ denote by $\varepsilon_i^{(1)}$ the class of the divisor which is the closure in $P_g$ of the locus of two-component curves having exactly the first marked point and other $i$ marked points on one of the two components. Then clearly $j^*(\delta_i) = \varepsilon_{i-1}^{(1)}$ for $i = 2, \ldots, g-2$. Moreover adapting the argument in [**EH89**, pg. 49], we have that

$$j^*(\delta_{g-1}) = -\sum_{i=1}^{g-3} \frac{i(g-i-1)}{g-2} \varepsilon_i^{(1)}$$

while

$$j^*(\delta_1) = -\sum_{i=1}^{g-3} \frac{(g-i-1)(g-i-2)}{(g-1)(g-2)} \varepsilon_i^{(1)}.$$

Finally since $j^*(\overline{\mathfrak{D}}^2_6) = 0$, checking the coefficient of $\varepsilon_i^{(1)}$ we obtain

$$b_{i+1} = \frac{(g-i-1)(g-i-2)}{(g-1)(g-2)} b_1 + \frac{i(g-i-1)}{g-2} b_{g-1}$$

for $i = 1, 2, 3$.

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# Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 21.12.2011 Nicola Tarasca