BRILL-NOETHER LOCI IN CODIMENSION TWO

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The classical Brill-Noether theory is a powerful tool for investigating subvarieties of moduli spaces of curves. While a general curve admits only linear series with non-negative Brill-Noether number, the locus $\mathcal{M}^r_{g,d}$ of curves of genus g admitting a $\mathfrak{g}^r_d$ with negative Brill-Noether number $\rho(g,r,d) := g-(r+1)(g-d+r) < 0$ is a proper subvariety of $\mathcal{M}_g$.

Such a locus can be realized as a degeneracy locus of a map of vector bundles over $\mathcal{M}_g$ so that one knows that the codimension of $\mathcal{M}^r_{g,d}$ is less than or equal to $-\rho(g,r,d)$ ([8]). When $\rho(g,r,d) \in \{-1,-2,-3\}$ the opposite inequality also holds (see [5] and [3]), hence the locus $\mathcal{M}^r_{g,d}$ is pure of codimension $-\rho(g,r,d)$. Moreover, the equality is classically known to hold also when $r=1$ and for any $\rho(g,1,d) < 0$: B. Segre first showed that the dimension of $\mathcal{M}^1_{g,d}$ is $2g+2d-5$, that is, $\mathcal{M}^1_{g,d}$ has codimension exactly $-\rho(g,1,d)$ for every $\rho(g,1,d)<0$ (see for instance [1]).

Harris, Mumford and Eisenbud have extensively studied the case $\rho(g,r,d)=-1$ when $\mathcal{M}^r_{g,d}$ is a divisor in $\mathcal{M}_g$ ([7], [4]). They computed the class of its closure in $\overline{\mathcal{M}}_g$ and found that it has slope $6+12/(g+1)$. Since for $g \geq 24$ this is less than $13/2$ the slope of the canonical bundle, it follows that $\overline{\mathcal{M}}_g$ is of general type for g composite and greater than or equal to 24.

While the class of the Brill-Noether divisor has served to reveal many important aspects of the geometry of $\overline{\mathcal{M}}_g$, very little is known about Brill-Noether loci of higher codimension. The main result presented in the talk is a closed formula for the class of the closure of the locus $\mathcal{M}^1_{2k,k} \subset \mathcal{M}_{2k}$ of curves of genus $2k$ admitting a pencil of degree k. Since $\rho(2k,1,k)=-2$, such a locus has codimension two. As an example, consider the hyperelliptic locus $\mathcal{M}^1_{4,2}$ in $\mathcal{M}_4$.

Faber and Pandharipande have shown that Hurwitz loci, in particular loci of type $\mathcal{M}^1_{g,d}$, are tautological in $\overline{\mathcal{M}}_g$ ([6]). When $g \geq 6$, Edidin has found a basis for the space $R^2(\overline{\mathcal{M}}_g, \mathbb{Q}) \subset A^2(\overline{\mathcal{M}}_g, \mathbb{Q})$ of codimension-two tautological classes of the moduli space of stable curves ([2]). It consists of the classes κ_1^2 and κ_2; the following products of classes from $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$: $\lambda\delta_0, \lambda\delta_1, \lambda\delta_2, \delta_0^2$ and δ_1^2; the following push-forwards $\lambda^{(i)}, \lambda^{(g-i)}, \omega^{(i)}$ and $\omega^{(g-i)}$ of the classes λ and $\omega=\psi$ respectively from $\overline{\mathcal{M}}_{i,1}$ and $\overline{\mathcal{M}}_{g-i,1}$ to $\Delta_i \subset \overline{\mathcal{M}}_g$: $\lambda^{(3)}, \ldots, \lambda^{(g-3)}$ and $\omega^{(2)}, \ldots, \omega^{(g-2)}$; finally the classes of closures of loci of curves having two nodes: the classes θ_i of the loci having as general element a union of a curve of genus i and a curve of genus $g-i-1$ attached at two points; the class δ_{00} of the locus whose general element is an irreducible curve with two nodes; the classes δ_{0j} of the closures of the loci of irreducible nodal curves of geometric genus $g-j-1$ with a tail of genus j; at last the classes δ_{ij} of the loci with general element a chain of three irreducible curves with the external ones having genus i and j.

Having then a basis for the classes of Brill-Noether codimension-two loci, in order to determine the coefficients I use the method of test surfaces. The idea is the following. Evaluating the intersections of a given a surface in $\overline{\mathcal{M}}_g$ on one hand with the classes in the basis and on the other hand with the Brill-Noether loci, one obtains a linear relation in the coefficients of the Brill-Noether classes. Hence

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one has to produce several surfaces giving enough independent relations in order to compute all the coefficients of the sought-for classes.

The surfaces used are bases of families of curves with several nodes, hence a good theory of degeneration of linear series is required. For this, the compactification of the Hurwitz scheme by the space of admissible covers introduced by Harris and Mumford comes into play. The intersection problems thus boil down first to counting pencils on the general curve, and then to evaluating the respective multiplicities via a local study of the compactified Hurwitz scheme.

Theorem ([9]). *For $k \geq 3$, the class of the locus $\overline{\mathcal{M}}^1_{2k,k} \subset \overline{\mathcal{M}}_{2k}$ is*

$$\left[\overline{\mathcal{M}}^1_{2k,k}\right]_Q = c\Bigg[A_{\kappa_1^2}\kappa_1^2 + A_{\kappa_2}\kappa_2 + A_{\delta_0^2}\delta_0^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\delta_1^2}\delta_1^2 + A_{\lambda\delta_1}\lambda\delta_1 + A_{\lambda\delta_2}\lambda\delta_2 + \sum_{i=2}^{2k-2} A_{\omega^{(i)}}\omega^{(i)} + \sum_{i=3}^{2k-3} A_{\lambda^{(i)}}\lambda^{(i)} + \sum_{i,j} A_{\delta_{ij}}\delta_{ij} + \sum_{i=1}^{\lfloor (2k-1)/2 \rfloor} A_{\theta_i}\theta_i\Bigg]$$

in $R^2(\overline{\mathcal{M}}_{2k}, \mathbb{Q})$, where

$$c = \frac{2^{k-6}(2k-7)!!}{3(k!)} \qquad A_{\kappa_1^2} = -A_{\delta_0^2} = 3k^2 + 3k + 5$$

$$A_{\kappa_2} = -24k(k+5) \qquad A_{\delta_1^2} = -(3k(9k+41)+5)$$

$$A_{\lambda\delta_0} = -24(3(k-1)k-5) \qquad A_{\lambda\delta_1} = 24\left(-33k^2+39k+65\right)$$

$$A_{\lambda\delta_2} = 24(3(37-23k)k+185) \qquad A_{\delta_{1,1}} = 48\left(19k^2-49k+30\right)$$

$$A_{\delta_{1,2k-2}} = \frac{2}{5}(3k(859k-2453)+2135) \qquad A_{\delta_{00}} = 24k(k-1)$$

$$A_{\delta_{0,2k-2}} = \frac{2}{5}(3k(187k-389)-745) \qquad A_{\delta_{0,2k-1}} = 2(k(31k-49)-65)$$

$$\begin{aligned}
A_{\omega^{(i)}} &= -180i^4 + 120i^3(6k+1) - 36i^2\left(20k^2+24k-5\right) \\
&\quad + 24i\left(52k^2-16k-5\right) + 27k^2 + 123k + 5 \\
A_{\lambda^{(i)}} &= 24\left(6i^2(3k+5) - 6i\left(6k^2+23k+5\right) + 159k^2 + 63k + 5\right) \\
A_{\theta^{(i)}} &= -12i\left(5i^3 + i^2(10-20k) + i\left(20k^2-8k-5\right) - 24k^2 + 32k - 10\right)
\end{aligned}$$

and for $i \geq 1$ and $2 \leq j \leq 2k-3$

$$A_{\delta_{ij}} = 2\left(3k^2(144ij-1) - 3k(72ij(i+j+4)+1) + 180i(i+1)j(j+1) - 5\right)$$

while

$$A_{\delta_{0j}} = 2\left(-3\left(12j^2+36j+1\right)k + (72j-3)k^2 - 5\right)$$

for $1 \leq j \leq 2k-3$.

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